Math 333 Problem Set 9
Solutions

Be sure to list EVERYONE in the that you talk to about the homework!

1. (a) Show that the set $\{(a,0) : a \in \mathbb{Z}\}$ is an ideal in $\mathbb{Z} \times \mathbb{Z}$.

   Proof. Set $I = \{(a,0) : a \in \mathbb{Z}\}$. Observe that $0_{\mathbb{Z}\times\mathbb{Z}} = (0,0) \in I$. Let $(a,0),(b,0) \in I$ and $(c,d) \in \mathbb{Z}\times\mathbb{Z}$. We have $(a,0)-(b,0) = (a-b,0) \in I$ and $(a,0)(c,d) = (ac,0) \in I$. Thus, I is an ideal in $\mathbb{Z}\times\mathbb{Z}$ as desired. $\square$

   (b) Show that the set $\{(a,a) : a \in \mathbb{Z}\}$ is not an ideal in $\mathbb{Z} \times \mathbb{Z}$.
   Set $J = \{(a,a) : a \in \mathbb{Z}\}$. Then $(1,1) \in J$ and $(1,2) \in \mathbb{Z}\times\mathbb{Z}$, but $(1,1)(1,2) = (1,2) \notin J$. Thus, J is not an ideal in $\mathbb{Z}\times\mathbb{Z}$.

2. Let R and S be rings and $I \subset R$, $J \subset S$ ideals. Show that $I \times J$ is an ideal in the ring $R \times S$.

   Proof. Since I and J are ideals, we have $0_R \in I$ and $0_S \in J$ so $(0_R, 0_S) \in I \times J$. Let $(a,b),(c,d) \in I \times J$ and $(r,s) \in R \times S$. We have $(a,b)-(c,d) = (a-c, b-d) \in I \times J$ because I and J are ideals. Similarly, we have $(r,s)(a,b) = (ra, sb) \in I \times J$ and $(a,b)(r,s) = (ar, bs) \in I \times J$ again because I and J are ideals. Thus, $I \times J$ is an ideal. $\square$

3. Show that if I is an ideal in a field F, then $I = \langle 0_F \rangle$ or $I = F$.

   Proof. If $I = \langle 0_R \rangle$ we are done, so assume there exists a nonzero element $a \in I$. Since $I \subset F$ and F is a field, we have a is a unit, i.e., there exists $b \in F$ so that $ab = 1_F$. The fact that I is an ideal gives $1_F = ab \in I$. Given any $r \in F$ we have $r = r1_F \in I$ and so $I = F$. $\square$

4. List all the distinct principal ideals in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

We have the principal ideals are given by $\langle([a]_2,[b]_3)\rangle$ for $a=0,1$ and $b=0,1,2$. The point is to see which of these is distinct. We have

$$\langle([0]_2,[0]_3)\rangle = \{([0]_2,[0]_3)\}$$
$$\langle([0]_2,[1]_2)\rangle = \{([0]_2,[0]_3),([0]_2,[1]_2),([0]_2,[2]_2)\}$$
$$\langle([0]_2,[2]_2)\rangle = \{([0]_2,[0]_3),([0]_2,[1]_2),([0]_2,[2]_2)\}$$
$$\langle([1]_2,[0]_3)\rangle = \{([0]_2,[0]_3),([1]_2,[0]_3)\}$$
$$\langle([1]_2,[1]_3)\rangle = \{([0]_2,[0]_3),([0]_2,[1]_3),([0]_2,[2]_3),([1]_2,[0]_3),([1]_2,[1]_3),([1]_2,[2]_3)\}$$
$$\langle([1]_2,[2]_3)\rangle = \{([0]_2,[0]_3),([0]_2,[1]_3),([0]_2,[2]_3),([1]_2,[0]_3),([1]_2,[1]_3),([1]_2,[2]_3)\}.$$

Thus, the distinct ideals are given by $\langle([0]_2,[0]_3)\rangle$, $\langle([0]_2,[1]_2)\rangle$, $\langle([1]_2,[0]_3)\rangle$, and $\langle([1]_2,[1]_3)\rangle$.

5. Let I be an ideal in R and S a subring of R. Prove that $I\cap S$ is an ideal in S.

Proof. Since I and S are both subrings, we have $0_R \in I\cap S$. Let $a,b\in I\cap S$. Since I and S are subrings we have $a-b\in I$ and $a-b\in S$, so $a-b\in I\cap S$. Let $s\in S$. Since S is a subring and $a,s\in S$, we have $as,sa\in S$. Since I is an ideal in R and $S\subset R$, we have $as,sa\in I$. Thus, $as,sa\in I\cap S$ and so $I\cap S$ is a subring of S. □

6. (a) Let I and J be ideals in a ring R. Define $I+J=\{i+j : i\in I, j\in J\}$. Show this is an ideal in R that contains I and J.

   Proof. Note that since I and J are ideals, $O_R\in I\cap J$ so $0_R = 0_R+0_R\in I+J$. Let $a,b\in I+J$, i.e., $a=i_1+j_1$ and $b=i_2+j_2$ for some $i_1,i_2\in I$, $j_1,j_2\in J$. We have $a-b=(i_1+j_1)-(i_2+j_2)=(i_1-i_2)+(j_1-j_2)\in I+J$. Let $r\in R$. Then $ra=r(i_1+j_1)=ri_1+rj_1\in I+J$ since I and J are ideals. Similarly, $ar\in I+J$. Thus, $I+J$ is an ideal in R. Moreover, given $i\in I$ we have $i=i+0_R\in I+J$ so $I\subset I+J$. Similarly, $J\subset I+J$. □

   (b) Let $a,b\in\mathbb{Z}$ and set $d=\gcd(a,b)$. Show that $\langle a\rangle+\langle b\rangle=\langle d\rangle$.

   Proof. Let $r\in\langle a\rangle+\langle b\rangle$, i.e., $r=ax+by$ for some $x,y\in\mathbb{Z}$. Since $d\mid a$ and $d\mid b$ there exists $s,t\in\mathbb{Z}$ so that $a=ds$ and $b=dt$. Thus, $r=ax+by=d(sx+ty)\in\langle d\rangle$. Thus, $\langle a\rangle+\langle b\rangle\subset\langle d\rangle$.

Let $z \in \langle d \rangle$, i.e., $z = df$ for some $f \in \mathbb{Z}$. Since $d = \gcd(a, b)$ there exists $m, n \in \mathbb{Z}$ so that $d = am + bn$. Thus, $z = df = (am + bn)f = a(mf) + b(nf) \in \langle a \rangle + \langle b \rangle$. Thus, $\langle d \rangle \subset \langle a \rangle + \langle b \rangle$. Combining this with the above containment gives equality. □

7. Let F be a field. Show that every ideal in the ring $F[x]$ is principal.

Proof. Let $I \subset F[x]$ be an ideal. If $I = \langle 0_F \rangle$ we are done, so assume $I \neq \langle 0_F \rangle$. Let $\mathcal{S} = \{f \in I : \deg f \geq 0\}$. This set is nonempty since $I \neq \langle 0_F \rangle$. Choose $g \in \mathcal{S}$ of minimal degree. We claim that $I = \langle g \rangle$. Clearly we have $\langle g \rangle \subset I$ since I is an ideal and $g \in I$. Let $h \in I$. Write $h = gq + r$ for $q, r \in F[x]$ with $r = 0_F$ or $\deg r < \deg g$. Observe that since $h \in I$ and $g \in I$ we have $r = h - gq \in I$. However, g has minimal degree in I so it must be the case that $r = 0_F$ and so $h \in \langle g \rangle$, i.e., $I = \langle g \rangle$ as claimed. Thus, every ideal in $F[x]$ is principal. □

8. (a) Prove that the set S of rational numbers (in lowest terms) with odd denominators is a subring of $\mathbb{Q}$.

Proof. We have $0 = 0/1 \in S$ clearly. Let $x, y \in S$, so $x = \frac{a}{b}$ and $y = \frac{c}{d}$ with b, d both odd. In particular, we know that bd is odd. We have $x - y = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. Moreover, $xy = \frac{ac}{bd} \in S$. Thus, S is a subring of $\mathbb{Q}$. □

(b) Let I be the set of elements in S with even numerators. Prove that I is an ideal in S.

Proof. We clearly have $0 = 0/1 \in I$. Let $x = \frac{a}{b}, y = \frac{c}{d} \in I$ and $z = \frac{e}{f} \in S$ be in lowest terms. We have $x - y = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. Observe since a and c are both even, so is $ad - bc$. Thus, $x - y \in I$. Moreover, we have $zx = \frac{e}{f}\frac{a}{b} = \frac{ea}{bf}$. Observe that x and z both have odd denominators, so xz has an odd denominator. Moreover, since x has an even numerator so does xz. Thus, $xz \in I$ and so I is an ideal of S. □

(c) Show the set S/I consists of exactly two distinct cosets.

Proof. Let $x = \frac{a}{b} \in S$. If a is even then $x \in I$ so $x + I = I$. Suppose that $x = \frac{a}{b}$ with a odd, say $a = 2k + 1$. Then we have $x + I = \frac{2k+1}{b} + I = \frac{2k}{b} + \frac{1}{b} + I = \frac{1}{b} + I$. Thus, $\frac{a}{b} \equiv \frac{1}{b} \pmod{I}$ if a is odd. Now observe that $\frac{1}{b} - \frac{1}{1} = \frac{1-b}{b}$. Since b is assumed to be odd, we have $1 - b$ is even and so $\frac{1}{b} - \frac{1}{1} \in I$, i.e., $\frac{1}{b} \equiv 1 \pmod{I}$. Thus, if $x = \frac{a}{b}$ with a odd we have $x + I = 1 + I$. Hence, $S/I = \{0 + I, 1 + I\}$. □