Math 333 Problem Set 5
Solutions

Be sure to list EVERYONE in the that you talk to about the homework!

1. Let $R = \left\{ \begin{bmatrix} a & c \\ 0 & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \right\}$ be a subset of $\mathrm{Mat}_2(\mathbb{Z})$. Is this a subring? Be sure to justify your answer.

Proof. We prove that R is a subring. First, note that R is not the empty set as it contains the matrix $0_{\mathrm{Mat}_2(\mathbb{Z})} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This also shows that R contains the additive identity of $\mathrm{Mat}_2(\mathbb{Z})$. Let $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $\begin{bmatrix} s & t \\ 0 & u \end{bmatrix}$ be elements of R. We have

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} s & t \\ 0 & u \end{bmatrix} = \begin{bmatrix} a+s & b+t \\ 0 & d+u \end{bmatrix} \in R$$

and

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} s & t \\ 0 & u \end{bmatrix} = \begin{bmatrix} as & at+bd \\ 0 & du \end{bmatrix} \in R.$$

Thus, R is closed under addition and multiplication. Observe that the additive inverse of $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ in R is $\begin{bmatrix} -a & -b \\ 0 & -d \end{bmatrix}$, which is clearly in R. Thus, R is a subring of $\mathrm{Mat}_2(\mathbb{Z})$. □

2. Write out addition and multiplication tables for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

+	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,0)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,1)	(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(0,2)	(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)
(1,0)	(1,0)	(1,1)	(1,2)	(0,0)	(0,1)	(0,2)
(1,1)	(1,1)	(1,2)	(1,0)	(0,1)	(0,2)	(0,0)
(1,2)	(1,2)	(1,0)	(1,1)	(0,2)	(0,0)	(0,1)

$\cdot$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,1)	(0,2)	(0,0)	(0,1)	(0,2)
(0,2)	(0,0)	(0,2)	(0,1)	(0,0)	(0, 2)	(0,1)
(1,0)	(0,0)	(0,0)	(0,0)	(1,0)	(1,0)	(1,0)
(1,1)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(1,2)	(0,0)	(0,2)	(0,1)	(1,0)	(1,2)	(1, 1)

3. Let R be a ring and r_0 a fixed element of R. Prove that $r_0R = \{r_0r : r \in R\}$ is a subring of R.

Proof. Observe that $0_R = r_0 0_R \in r_0R$ so r_0R is nonempty and contains the additive identity. Let r_0r and r_0s be elements of r_0R. We have $r_0r + r_0s = r_0(r + s) \in r_0R$ and $(r_0r)(r_0s) = r_0(rr_0s) \in r_0R$. Thus, r_0R is closed under addition and multiplication. Observe that since $r \in R$, there is a solution, say a, to the equation $r+x = 0_R$. This gives that $r_0r + r_0a = r_0 0_R = 0_R$. Since $r_0a \in r_0R$, we have the equation $r_0r+x = 0_R$ has a solution in r_0R and so r_0R is closed under additive inverses. Thus, r_0R is a subring of R. □

4. Define a new addition and multiplication on $\mathbb{Z}$ by setting $a \oplus b = a+b-1$ and $a \odot b = a + b - ab$ for all $a, b \in \mathbb{Z}$ where the operations on right hand side of the definitions are the usual ones in $\mathbb{Z}$. Prove that with these new operations $\mathbb{Z}$ is an integral domain.

Proof. Let R denote the set $\mathbb{Z}$ with these new operations. We clearly have R is not the empty set, so we now go through the eight requirements to be a ring. We list this numerically to ease the proof. Let $a, b, c \in R$.

(1) & (6) Since $a \oplus b = a + b - 1$ and $a \odot b = a + b - ab$, we have $a \oplus b \in R$ and $a \odot b \in R$ clearly.

(2) We have

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus (b + c - 1) \\ &= a + (b + c - 1) - 1 \\ &= (a + b - 1) + c - 1 \\ &= (a + b - 1) \oplus c \\ &= (a \oplus b) \oplus c. \end{aligned}$$

(3) We have

$$\begin{aligned} a \oplus b &= a + b - 1 \\ &= b + a - 1 \\ &= b \oplus a. \end{aligned}$$

(4) Set $0_R = 1$. We have

$$\begin{aligned} a \oplus 0_R &= a \oplus 1 \\ &= a + 1 - 1 \\ &= a \\ &= (1 + a) - 1 \\ &= (1 \oplus a) \\ &= 0_R \oplus a. \end{aligned}$$

(5) Observe we have

$$\begin{aligned} a \oplus (2 - a) &= a + (2 - a) - 1 \\ &= 1 \\ &= 0_R. \end{aligned}$$

Thus, the equation $a \oplus x = 0_R$ has a solution for each $a \in R$.

(7) Observe we have

$$\begin{aligned} a \odot (b \odot c) &= a \odot (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \end{aligned}$$

and

$$\begin{aligned} (a \odot b) \odot c &= (a + b - ab) \odot c \\ &= a + b - ab + c - (a + b - ab)c \\ &= a + b + c - ab - ac - ab + abc \end{aligned}$$

As these two are equal, we get associativity of multiplication.

(8) Observe we have

$$\begin{aligned} a \odot (b \oplus c) &= a \odot (b + c - 1) \\ &= a + b + c - 1 - a(b + c - 1) \\ &= a + b + c - 1 - ab - ac + a \\ &= 2a + b + c - 1 - ab - ac \end{aligned}$$

and

$$\begin{aligned} (a \odot b) \oplus (a \odot c) &= (a + b - ab) \oplus (a + c - ac) \\ &= a + b - ab + a + c - ac - 1 \\ &= 2a + b + c - 1 - ab - ac. \end{aligned}$$

Thus, $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$. Similarly one shows that $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.

Finally, we must show R is an integral domain, i.e., it is commutative with identity and if $a \odot b = 0_R$ then $a = 0_R$ or $b = 0_R$. It is clear that $a \odot b = b \odot a$. Set $1_R = 0$. Then for any $a \in R$ we have $a \odot 1_R = a + 0 - a(0) = a$, so $1_R = 0$ is the identity element of R. Let $a \odot b = 0_R$, i.e., $a + b - ab = 1$. This gives $(a-1)(b-1) = 0$. Since this equality is in the regular integers, we must have $a = 1$ or $b = 1$, i.e., $a = 0_R$ or $b = 0_R$ as required. □

5. Prove that $S = \{0, 2, 4, 6, 8\}$ is a subring of $\mathbb{Z}/10\mathbb{Z}$. Does S have an identity?

Proof. Clearly we have S is not the empty set. We can check it is a subring from the addition and multiplication tables given here:

+	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

·	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

We see from these that S is closed under addition and multiplication and each element has an additive inverse since there is a 0 in each row of the addition table. Thus, S is a subring of $\mathbb{Z}/10\mathbb{Z}$. Finally, the multiplication table shows that 6 is the multiplicative identity of this subring. $\square$

6. Let $S = \{a, b, c\}$ and $P(S)$ the set of all subsets of S. Define addition and multiplication on $P(S)$ by setting $M + N = (M - N) \cup (N - M)$ and $MN = M \cap N$. Write out addition and multiplication tables for $P(S)$.

Let $O = \emptyset$, $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{a, b\}$, $E = \{a, c\}$, $F = \{b, c\}$. We have

+	0	A	B	C	D	E	F	S
0	0	A	B	C	D	E	F	S
A	A	0	D	E	B	C	S	F
B	B	D	0	F	A	S	C	E
C	C	E	F	0	S	A	B	D
D	D	B	A	S	0	F	E	C
E	E	C	S	A	F	0	D	B
F	F	S	C	B	E	D	0	A
S	S	F	E	D	C	B	A	0

·	0	A	B	C	D	E	F	S
0	0	0	0	0	0	0	0	0
A	0	A	0	0	A	A	0	A
B	0	0	B	0	B	0	B	B
C	0	0	0	C	0	C	C	C
D	0	A	B	0	D	A	B	D
E	0	A	0	C	A	E	C	E
F	0	0	B	C	B	C	F	F
S	0	A	B	C	D	E	F	S