Math 333 Problem Set 2
Due: 02/17/16

Solutions

1. Find the quotient and remainder when $a = -614$ is divided by $b = 13$.

   We have $-614 = 13(-48) + 10$, so $q = -48$ and $r = 10$.

2. Prove that the square of any integer a is either of the form $3k$ or $3k+1$ for some integer k.

   Let $a \in \mathbb{Z}$. The division algorithm allows us to write $a = 3q + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < 3$. If $r = 0$, then $a^2 = 9q^2 = 3(3q^2)$. If $r = 1$, then $a^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$. If $r = 2$, then $a^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1$. Thus, we have a^2 is of the form $3k$ or $3k + 1$ in every possible case.

3. Use the division algorithm to prove that every odd integer is of the form $4k + 1$ or $4k + 3$ for some integer k.

   Let $n \in \mathbb{Z}$ be an odd integer. The division algorithm allows us to write $n = 4q + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < 4$. Observe that if $r = 0$ then $n = 4q$ is even and if $r = 2$ we have $n = 4q + 2$, which is even. Thus, if n is odd we must have $n = 4q + 1$ or $n = 4q + 3$, as claimed.

4. If $a \mid b$ and $b \mid c$, prove that $a \mid c$.

   Since $a \mid b$ there exists $m \in \mathbb{Z}$ so that $b = am$. Similarly, since $b \mid c$ there exists $n \in \mathbb{Z}$ so that $c = bn$. Thus, $c = bn = (am)n = a(mn)$. Thus, $a \mid c$.

5. If $a \mid b$ and $a \mid c$ prove that $a \mid (bm + cn)$ for all integers $m, n \in \mathbb{Z}$.

   Since $a \mid b$ there exists $s \in \mathbb{Z}$ so that $b = as$ and since $a \mid c$ there exists $t \in \mathbb{Z}$ so that $c = bt$. Thus, we have $bm + cn = (as)m + (at)n =$

$a(sm + tn)$. Hence $a \mid (bm + cn)$.

6. If $a \mid c$ and $b \mid c$, does $ab \mid c$? Be sure to justify your answer.

   This is a false statement. For instance, take $a = b = c = 2$. Then $2 \mid 2$ but $2 \cdot 2 = 4 \nmid 2$.

7. Prove that $\gcd(n, n+1) = 1$ for all $n \in \mathbb{Z}$.

   Let $d = \gcd(n, n+1)$. Then $d > 0$ and $d \mid ((n+1) + (-1)n) = 1$ by problem 5. Thus, we must have $d = 1$.