Math 333 Problem Set 1
Solutions

Be sure to list EVERYONE in the class that you talk to about the homework!

1. Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, prove $A \subseteq C$.

   Let $a \in A$. The fact that $A \subseteq B$ implies that $a \in B$. Now, since $B \subseteq C$ we have $a \in C$ as well. Since every element of A is contained in C, we must have $A \subseteq C$.

2. Let A, B and C be sets. Prove that

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

   Let $(a, x) \in A \times (B \cup C)$. This means that $a \in A$ and $x \in B \cup C$, i.e., $x \in B$ or $x \in C$. If $x \in B$, then $(a, x) \in A \times B$ and if $x \in C$ then $(a, x) \in A \times C$. In either case, we see $(a, x) \in (A \times B) \cup (A \times C)$ and so $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

   Now let $(a, x) \in (A \times B) \cup (A \times C)$, i.e., $(a, x) \in A \times B$ or $(a, x) \in A \times C$. If $(a, x) \in A \times B$ then $a \in A$ and $x \in B$ so $(a, x) \in A \times (B \cup C)$. If $(a, x) \in A \times C$ then $a \in A$ and $x \in C$ so $(a, x) \in A \times (B \cup C)$. Thus, we have $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

   Since we have shown containment in each direction we must have equality of the sets.

3. Give an example of a function that is injective but not surjective and an example of a function that is surjective but not injective. Be sure to prove your examples are correct.

   Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Define a function $f : A \to B$ by setting $f(1) = a$ and $f(2) = b$. Clearly it is injective. It is not surjective because c is not in the image of f.

   Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Define a function $f : A \to B$ by setting $f(1) = f(2) = a$ and $f(3) = b$. Since a and b are both in the image of f the function is surjective. It is not injective because

$f(1) = f(2)$ but $1 \neq 2$.

4. (a) Prove the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n + 4$ is injective.

   Let $a, b \in \mathbb{Z}$ so that $f(a) = f(b)$. Then we have $2a + 4 = 2b + 4$. Subtracting 4 from each side and then dividing by 2 we obtain $a = b$. Thus, f is injective.

   (b) Prove the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(x) = \ln(x)$ is surjective.

   Let $b \in \mathbb{R}$. Then we have $e^b \in \mathbb{R}_{>0}$ and $f(e^b) = \ln(e^b) = b$, so f is surjective.

5. Let $f : A \to B$ and $g : B \to C$ be functions.

   (a) Prove that if f and g are injective, then $g \circ f$ is injective.

   Let $a_1, a_2 \in A$ so that $(g \circ f)(a_1) = (g \circ f)(a_2)$, i.e., $g(f(a_1)) = g(f(a_2))$. Since g is injective we must have $f(a_1) = f(a_2)$. Now we use that f is injective to obtain $a_1 = a_2$, i.e., $g \circ f$ is injective.

   (b) Prove that if f and g are surjective, then $g \circ f$ is surjective.

   Let $c \in C$. Since g is surjective there exists $b \in B$ so that $g(b) = c$. We now use that f is injective to see there exists $a \in A$ so that $f(a) = b$. Observe that $(g \circ f)(a) = g(f(a)) = g(b) = c$. Thus, $g \circ f$ is surjective.

6. Prove that for each $n \in \mathbb{Z}_{\geq 0}$ one has $2^n > n$.

   We prove this by induction on n. Our base case is when $n = 1$. In this case we clearly have $2 > 1$ so the base case holds. Our induction hypothesis is that $2^k > k$ for some $k \in \mathbb{Z}_{\geq 1}$. We have

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k \quad \text{by the induction hypothesis} \\ &= k + k \\ &\geq k + 1 \end{aligned}$$

where the last inequality follows because $k \in \mathbb{Z}_{\geq 1}$. Thus, by induction we see the statement is true for all $n \in \mathbb{Z}_{\geq 1}$.

7. Prove that 5 is a factor of $2^{4n-2} + 1$ for every $n \in \mathbb{Z}_{>0}$.

We prove this by induction on n. The base case is $n = 1$. In this case we have $2^{4(1)-2} + 1 = 5$, which is clearly divisible by 5 so the base case is true. Our induction hypothesis is that 5 divides $2^{4k-2} + 1$ for some $k \in \mathbb{Z}_{>0}$. We have

$$\begin{aligned} 2^{4(k+1)-2} + 1 &= 2^{4k+4-2} + 1 \\ &= 2^4 \cdot 2^{4k-2} + 1 \\ &= 2^4 \cdot 2^{4k-2} + 2^4 - 2^4 + 1 \\ &= 2^4(2^{4k-2} + 1) - 15. \end{aligned}$$

Note 5 divides the first term by our induction hypothesis and clearly divides 15, so it divides their difference. Thus, by induction our statement holds for all $n \in \mathbb{Z}_{>0}$.