Math 333 Problem Set 10
Due: 05/11/16

Be sure to list EVERYONE in the that you talk to about the homework!

1. (a) Prove the set T of matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ with $a, b \in \mathbb{R}$ is a subring of $\mathrm{Mat}_2(\mathbb{R})$.

   Proof. Note that the zero matrix is in T clearly. Let $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \in T$. Then we have

   $$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} - \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ 0 & a-c \end{bmatrix} \in T,$$

   $$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix} \in T,$$

   and

   $$\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix} \in T.$$

   Thus, T is a subring of $\mathrm{Mat}_2(\mathbb{R})$. □

   (b) Prove the set I of matrices of the form $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ with $b \in \mathbb{R}$ is an ideal in the ring T.

   Proof. We clearly have the zero matrix is in I. Let $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} \in I$ and $\begin{bmatrix} e & f \\ 0 & e \end{bmatrix} \in T$. Then we have

   $$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b-d \\ 0 & 0 \end{bmatrix} \in I$$

   and

   $$\begin{bmatrix} e & f \\ 0 & e \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & eb \\ 0 & 0 \end{bmatrix} \in I.$$

   Note that we saw in part (a) that the multiplication in T is commutative so we only need to check one-sided multiplication here as well. Thus I is an ideal in T. □

(c) Show that every coset in T/I can be written in the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + I$.

Proof. Let $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} + I \in T/I$. We have

$$\begin{aligned} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} + I &= \left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right) + I \\ &= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + I \right) \\ &= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + I. \end{aligned}$$

$\square$

2. Let R be a ring. Show that the map $\varphi : R[x] \to R$ that sends each polynomial to its constant term is a surjective ring homomorphism.

Proof. Let $r \in R$. Since $R \subset R[x]$, we have $r \in R[x]$ and $\varphi(r) = r$. Thus the map is surjective. Let $f = \sum_{i=0}^{m} a_j x^i$, $g = \sum_{j=0}^{n} x^j \in R[x]$. Without loss of generality assume $m \leq n$. Define $a_{m+1} = a_{m+2} = \cdots = a_n = 0$ so we can write $f = \sum_{i=0}^{n} a_i x^i$. We have

$$\begin{aligned} \varphi(f+g) &= \varphi\left(\sum_{j=0}^{n} (a_j + b_j) x^j \right) \\ &= a_0 + b_0 \\ &= \varphi(f) + \varphi(g). \end{aligned}$$

Recall that $fg = \sum_{j=0}^{m+n} c_j x^j$ where $c_j = \sum_{k=0}^{j} a_k b_{j-k}$. Note the constant term here is $c_0 = a_0 b_0$. This gives

$$\begin{aligned} \varphi(fg) &= a_0 b_0 \\ &= \varphi(f)\varphi(g). \end{aligned}$$

Thus, φ is a surjective ring homomorphism. $\square$

3. Let F be a field, R a nonzero ring, and $\varphi : F \to R$ a surjective ring homomorphism. Prove that φ is an isomorphism.

Proof. Let $K = \ker \varphi$. Since K is an ideal of F we know from the previous homework set that $K = \{0_F\}$ or $K = F$. If $K = \{0_F\}$ we are done. Suppose $K = F$. This gives $\varphi(F) = \{0_R\}$. However, since φ is surjective we must have $\varphi(F) = R$. This contradicts R being a nonzero ring so it must be that $K = \{0_F\}$. $\square$

4. (a) Let $\varphi : R \to S$ be a surjective homomorphism of rings. Let I be an ideal in R. Prove that $\varphi(I)$ is an ideal in S.

   Proof. We have $0_S \in \varphi(I)$ because $0_R \in I$ as I is an ideal and we know $\varphi(0_R) = 0_S$ because φ is a ring homomorphism. Let $s_1, s_2 \in \varphi(I)$ and $s \in S$. Since φ is surjective there exists $r_1, r_2 \in I$ and $r \in R$ so that $\varphi(r_1) = s_1, \varphi(r_2) = s_2$, and $\varphi(r) = s$. We have $s_1 - s_2 = \varphi(r_1) - \varphi(r_2) = \varphi(r_1 - r_2)$ so $s_1 - s_2 \in \varphi(I)$ as $r_1 - r_2 \in I$. Similarly, we have $ss_1 = \varphi(r)\varphi(r_1) = \varphi(rr_1) \in \varphi(I)$ and $s_1 s = \varphi(r_1)\varphi(r) = \varphi(r_1 r) \in \varphi(I)$ using that $rr_1, r_1 r \in I$ because I is an ideal. Thus, $\varphi(I)$ is an ideal. $\square$

   (b) Is part (a) true if φ is not surjective? Prove it is true or give a counterexample.

   This is not true if φ is not surjective. Consider the map $\varphi : \mathbb{Z} \to \mathbb{Q}$ given by $\varphi(n) = n/1$. It is straightforward to check this is a homomorphism. We have $\mathbb{Z}$ is certainly an ideal in $\mathbb{Z}$, but $\varphi(\mathbb{Z}) = \mathbb{Z}$, which we have already seen in class is not an ideal in $\mathbb{Q}$.

5. Let I be an ideal in a ring R. Prove that every element in R/I has a square root if and only if for every $r \in R$ there exists $a \in R$ so that $r - a^2 \in I$.

Proof. Let $r + I \in R/I$ and suppose $r + I$ has a square root, i.e, there exists $a + I \in R/I$ so that $(a + I)^2 = r + I$. We can rewrite this as $a^2 + I = r + I$, i.e., $r - a^2 \in I$. Now suppose for every $r \in R$ there exists $a \in R$ so that $r - a^2 \in I$. Equivalently we have $(r - a^2) + I = 0_R + I$, i.e, $r + I = a^2 + I = (a + I)^2$. Thus, $r + I$ has a square-root. $\square$

6. (a) Let I and K be ideals in a ring R with $K \subset I$. Prove that $I/K = \{a + K : a \in I\}$ is an ideal in the quotient ring R/K.

Proof. Since I is an ideal we have $0_R \in I$, so $0_{R/K} = 0_R + K \in I/K$. Let $a + K, b + K \in I/K$ and $r + K \in R/K$. We have $(a+K)-(b+K) = (a-b)+K \in I/K$ since $a-b \in I$. Moreover, since I is an ideal we have $ra \in I$ and $ar \in I$ so $(r+K)(a+K) = ra + K \in I/K$ and $(a+K)(r+K) = ar + K \in I/K$. Thus, I/K is an ideal in R/K. □

(b) Prove that $(R/K)/(I/K) \cong R/I$. (Hint: Define a map $\varphi : R/K \to R/I$ given by $\varphi(r + K) = r + I$. Show this is well-defined, a surjective ring homomorphism, and find its kernel.)

Proof. Define $\varphi : R/K \to R/I$ by $\varphi(r + K) = r + I$. We first must show this map is well-defined. Let $r_1 + K = r_2 + K$ in R/K, i.e., $r_1 = r_2 + k$ for some $k \in K$. This gives

$$\begin{aligned} \varphi(r_1 + K) &= r_1 + I \\ &= (r_2 + k) + I \\ &= r_2 + I \quad (\text{since } k \in K \subset I) \\ &= \varphi(r_2 + K). \end{aligned}$$

Thus, φ is well-defined.
Let $r + I \in R/I$. We have $\varphi(r + K) = r + I$ so φ is surjective.
Let $r_1 + I, r_2 + I \in R/I$. Then

$$\begin{aligned} \varphi(r_1 + K + r_2 + K) &= \varphi((r_1 + r_2) + K) \\ &= (r_1 + r_2) + I \\ &= r_1 + I + r_2 + I \\ &\varphi(r_1 + K) + \varphi(r_2 + K) \end{aligned}$$

and

$$\begin{aligned} \varphi((r_1 + K)(r_2 + K)) &= \varphi((r_1 r_2) + K) \\ &= (r_1 r_2) + I \\ &= (r_1 + I)(r_2 + I) \\ &\varphi(r_1 + K)\varphi(r_2 + K). \end{aligned}$$

Thus, φ is a surjective ring homomorphism. To complete the proof it remains to show that $\ker\varphi = I/K$. Let $r + I \in \ker\varphi$. Then we have $0_R + I = \varphi(r + K) = r + I$. Thus, $r \in I$ and so $r + K \in I/K$. This gives $\ker\varphi \subset I/K$. Let $i + K \in I/K$. Then we have $\varphi(i + K) = i + I = 0_R + I$, i.e., $i + K \in \ker\varphi$. This gives $I/K = \ker\varphi$ and concludes the proof. $\square$