Chapter 0 : Review of Background Material

Hopefully everyone has some experience with proofs, sets, functions, and induction. We will briefly review these concepts, but for those that need a more in depth review please see the appendices of Hungerford.

0.1 Logic and Proof:

Logic and proof form the basis of mathematics. One has to become comfortable with these to do well in mathematics. We review the basics briefly.

Let P and Q be statements. For instance P could be the statement "it is February" and Q the statement "it is raining outside."

"P and Q" means both are true

"P or Q" means at least one of P and Q is true.

The negation of P is "it is not the case it is P". For instance, it would be "it is not February". Negation is more complicated when "and" and "or" are involved. The negation of "P and Q" is "not P or not Q". So one at least one is false.

For our example it would be "it is not Feb" or "it is not raining".

<u>In groups:</u> Negate

- "Everyone will get an A in physics" (labelled P)

- "Everyone will get an A in physics" (labelled P) and "Pat will get at least a 76 on the calculus final" (labelled Q).

How about "P or Q"? This negates as "not P and not Q".

How would "P or Q" above negate?

Quantifiers and their properties are extremely important but take practice to get used to. The <u>universal quantifier</u> says something is true for every object under consideration.

- Every day ends in a y
- For every nonzero real number x, $|x| > 0$.

The <u>existential quantifier</u> says there exists at least one object with a property.

- There is ~~at least~~ a student happy to be reviewing this material.

- There is a prime number between 6 and 8.

The negation of a statement with a universal quantifier is a statement with an existential quantifier.

P: The Golden State warriors will win every game.

~P: There exist a game the Golden state warriors will lose.

The negation of a statement with an existential quantifier is a statement with a universal quantifier.

P: There exists a class more fun than abstract algebra.

~P: Every class is less fun than abstract algebra.

Conditional statements are of the form "If P, then Q". This means if P is true, then Q is true as well. The contrapositive of this is "If not Q, then not P.". These are equivalent statements, though one may be easier to work with in a proof than the other.

- If n is even and $n>3$, then n is not prime.

This is of the form If P and Q, then R. Then contrapositive is If ~R, then ~(P and Q). So it is:

- If n is prime, then n is odd or $n \leq 3$.

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. These are NOT the same.

- If $p>2$ is prime, then p is odd. The converse is "If p is odd, then $p>2$ is prime. This is NOT true!

We say P iff Q if $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true!

Please read "methods of proof" in appendix A.

0.2 Sets and Functions:

A set is any collection of objects.

We write $A \subseteq B$ if $x \in A \Rightarrow x \in B$. We write $A \subsetneq B$ if $A \subseteq B$ and there is an element of B not in A. If $A \subseteq B$ and $B \subseteq A$ we write $A = B$.

$A \setminus B = \{ x : x \in A, x \notin B \}$

$A \cup B = \{ x : x \in A \text{ or } x \in B \}$

$A \cap B = \{ x : x \in A \text{ and } x \in B \}$.

$A \times B = \{ (x, y) : x \in A, y \in B \}$.

Given sets A and B, a function from A to B, written $f: A \to B$ or

$A \xrightarrow{f} B$, is a rule that assign exactly one element of B to each

elements of A.

- $f: \mathbb{R} \to \mathbb{R} \qquad x \mapsto x^2$

- $f: \{\text{students in this class}\} \to \{A, B, C, D, F\}$ by assigning final grades.

Given $f: A \to B$ and $g: B \to C$, we can compose them:

$$g \circ f : A \to C \quad \text{by}$$

$$x \in A \mapsto f(x) \in B \mapsto g(f(x)) \in C.$$

<u>Example:</u> $f: \mathbb{Z} \to \mathbb{Z} \qquad n \mapsto n+5$

$g: \mathbb{Z} \to \mathbb{Z} \qquad m \mapsto m^3.$

$$(f \circ g)(m) = f(m^3) = m^3 + 5$$

$$(g \circ f)(n) = g(n+5) = (n+5)^3.$$

We say a function $f: A \to B$ is <u>injective</u> or <u>1-1</u> if whenever

$f(a_1) = f(a_2)$, we must have $a_1 = a_2$.

(This is the "horizontal line test" from calculus!)

Example: 1) $f: \mathbb{Z} \to \mathbb{Z}$ $f(n) = n+2$ is injective because

if $f(n_1) = f(n_2)$, then $n_1 + 2 = n_2 + 2$, ie, $n_1 = n_2$.

2) $f: \mathbb{Z} \to \mathbb{Z}$ $f(n) = n^2$ is not injective because

$f(-1) = f(1)$ but $-1 \neq 1$.

Note to prove something you must show it is always true; to show something is false you just need one counterexample!

A function $f: A \to B$ is surjective or onto if for every $b \in B$ there exists an $a \in A$ so that $f(a) = b$.

Example: $f: \mathbb{R} \to \mathbb{R}$ $x \mapsto x^2$ is not surjective because -1 is not in the image. However, $g: \mathbb{R} \to \mathbb{R}_{\geq 0}$ $x \mapsto x^2$ is surjective because given $b \in \mathbb{R}_{\geq 0}$, $\sqrt{b} \in \mathbb{R}$ and $g(\sqrt{b}) = b$.

We say a function is bijective if it is injective and surjective.

Example: 1) Define $f: \mathbb{Z} \to \mathbb{Z}$ by $n \mapsto n+2$. ~~Then~~ This is bijective. (Be sure you can prove it!)

2) Let $2\mathbb{Z} = \{ n \in \mathbb{Z} : 2 | n \}$. Define $f: \mathbb{Z} \to 2\mathbb{Z}$ by
$n \mapsto 2n$.

We claim this is a bijection. First, suppose $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{Z}$. Then $2n_1 = 2n_2$, ie, $n_1 = n_2$. Thus, f is injective. Now let $n \in 2\mathbb{Z}$. Then there is an $m \in \mathbb{Z}$ so that $n = 2m$. We have $f(m) = n$, so f is surjective as well.

Thm: If $f: A \to B$ is bijective then there exists a function $f^{-1}: B \to A$ so that $f \circ f^{-1}(b) = b$ for all $b \in B$ and $f^{-1} \circ f(a) = a$ for every $a \in A$. If such f^{-1} exist, then f is bijective. (f is bijective if and only if it has an inverse.)

Proof: Suppose f is bijective. We need to define $f^{-1}: B \to A$. Let $b \in B$. Since f is surjective $\exists\, a \in A$ so that $f(a) = b$. Define $f^{-1}(b) = a$. We need to see this is well-defined. Note since f is injective a is unique so this is well-defined. This defines a map. Note we have by construction that $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = f^{-1}(b) = a$. Thus, f^{-1} is the inverse.

Now suppose f^{-1} exists. Let $b \in B$. Then $f^{-1}(b) = a \in A$ for some $a \in A$, so $f(a) = f(f^{-1}(b)) = b$ and so f is surjective.

Suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$. Then

$$a_1 = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = a_2,$$ so f is injective. ∎

0.3 Well-ordering and induction:

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\} = \mathbb{Z}_{\geq 0}$. We assume the following axiom.

Well-ordering axiom: Every nonempty subset of $\mathbb{Z}_{\geq 0}$ has a smallest element.

Theorem (Complete Induction): Assume that for each $n \in \mathbb{Z}_{\geq r}$ a statement $P(n)$ is given. If

$r \in \mathbb{Z}_{\geq 0}$

(i) $P(r)$ is a true statement;

and

(ii) Whenever $P(j)$ is true for all j s.t $r \leq j < t$, then $P(t)$ is true,

Then $P(n)$ is true for all $n \in \mathbb{Z}_{\geq r}$.

Example: Prove that for any real number $r \neq 1$, and all $n \in \mathbb{Z}_{\geq 1}$

$$1+r+r^2+\cdots+r^{n-1} = \frac{r^n-1}{r-1}.$$

<u>Proof:</u> The statement $P(n)$ is:

$$P(n): \qquad 1+r+r^2+\cdots+r^{n-1} = \frac{r^n-1}{r-1}.$$

Our base case is $P(1)$: $1 = \frac{r-1}{r-1}$. This is clearly true.

Induction hypothesis: Assume that $P(k)$ is true for all $1 \le k \le N$, i.e., $1+r+r^2+\cdots+r^{k-1} = \frac{r^k-1}{r-1}$ for all $1 \le k \le N$.

We must show $P(N+1)$ is true. We have

$$1+r+r^2+\cdots+r^{N-1}+r^N = (1+r+r^2+\cdots+r^{N-1})+r^N$$

$$= \frac{r^N-1}{r-1} + r^N \qquad \text{by I.H.}$$

$$= \frac{r^N-1}{r-1} + \frac{(r-1)r^N}{r-1} = \frac{r^N-1+r^{N+1}-r^N}{r-1}$$

$$= \frac{r^{N+1}-1}{r-1}.$$

Thus, we have $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 1}$ by induction. ∎