MATH 142 — MIDTERM 1

February 29, 2016

NAME: Solutions

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 7 problems.
3. Write your name on the top of EVERY sheet of the exam at the start of the exam!
4. If you separate pages of this exam and include additional pages, please be sure to staple them in the correct order before turning the exam in.
5. Part of what you are being tested on is your ability to interpret the problem. Therefore I will not answer questions during the exam such as "am I doing this correctly?" or "what are you asking here?".
6. You may use a TI-84 or lower level calculator on the exam, but be careful to be precise on what is equal and what is an approximation.

PROBLEM	POINTS	SCORE
1	15	
2	10	
3	30	
4	10	
5	10	
6	10	
7	15	
TOTAL	100	

1. (5 points each) Let $f(x) = \dfrac{x^2}{x+1}$.

(a) Approximate the integral $\int_1^5 f(x)dx$ with a left hand sum using 4 intervals.

Since we want to use 4 intervals we have $\Delta x = \frac{5-1}{4} = 1$. Thus we have

$$\begin{aligned}\int_1^5 f(x)dx &\approx (f(1) + f(2) + f(3) + f(4)) \cdot 1 \\ &= \frac{1^2}{1+1} + \frac{2^2}{2+1} + \frac{3^2}{3+1} + \frac{4^2}{4+1} \\ &= \frac{437}{60}.\end{aligned}$$

(b) Approximate the integral $\int_1^5 f(x)dx$ with a right hand sum using 4 intervals.

We have the same Δx as in part (a). Thus,

$$\begin{aligned}\int_1^5 f(x)dx &\approx (f(2) + f(3) + f(4) + f(5)) \cdot 1 \\ &= \frac{2^2}{2+1} + \frac{3^2}{3+1} + \frac{4^2}{4+1} + \frac{5^2}{5+1} \\ &= \frac{219}{20}.\end{aligned}$$

(c) Approximate the integral $\int_1^5 f(x)dx$ using midpoint rule using 4 intervals.

The Δx is again the same, but now we need to take the midpoints of the intervals as our evaluation points. We have

$$\begin{aligned}\int_1^5 f(x)dx &\approx (f(1.5) + f(2.5) + f(3.5) + f(4.5)) \cdot 1 \\ &= \frac{1.5^2}{1.5+1} + \frac{2.5^2}{2.5+1} + \frac{3.5^2}{3.5+1} + \frac{4.5^2}{4.5+1} \\ &= \frac{5423}{630}.\end{aligned}$$

2. (10 points) Let $\int_1^{10} f(x)dx = -3$ and $\int_1^{10} g(x)dx = 6$. Find $\int_1^{10} [4f(x) - 10g(x)]dx$.

We have

$$\begin{aligned}\int_1^{10} [4f(x) - 10g(x)]dx &= 4\int_1^{10} f(x)dx - 10\int_1^{10} g(x)dx \\ &= 4(-3) - 10(6) \\ &= -72.\end{aligned}$$

3. (6+8+8+8points) **(a)** Fill in the blanks: The fundamental theorem of calculus states that if $F'(x)$ is a continuous function on $[a, b]$, then

$$\int_a^b \underline{F'(x)}dx = \underline{F(b) - F(a)}.$$

(b) Evaluate $\int_{-1}^{3}(x^2 + 2x + 1)dx$.

Observe that if $F(x) = \frac{x^3}{3} + x^2 + x$, then $F'(x) = x^2 + 2x + 1$. Thus, the fundamental theorem gives

$$\begin{aligned}\int_{-1}^{3}(x^2 + 2x + 1)dx &= F(3) - F(-1)\\ &= \frac{64}{3}.\end{aligned}$$

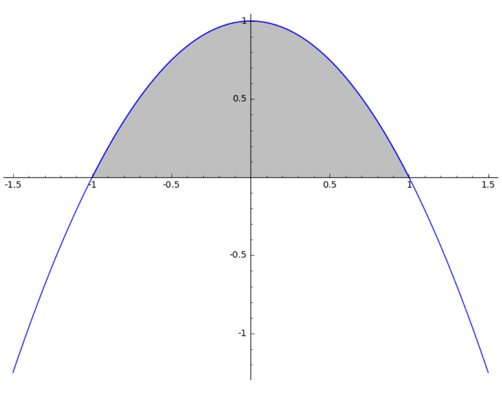
(c) Evaluate $\int_1^4 \sqrt{y}(2 + y)dy$.

First note that $\sqrt{y}(2 + y) = 2\sqrt{y} + y^{3/2}$. Now observe that if $F(y) = \frac{4}{3}y^{3/2} + \frac{2}{5}y^{5/2}$, then $F'(y) = 2y^{1/2} + y^{3/2}$. Thus, the fundamental theorem gives

$$\begin{aligned}\int_1^4 \sqrt{y}(2 + y)dy &= F(4) - F(1)\\ &= \frac{326}{15}.\end{aligned}$$

(d) Calculate the area of the region that lies under the curve $y = 1 - x^2$ and above the x-axis.

The region is given in the following picture:



This shows the area is given by $\int_{-1}^{1}(1 - x^2)dx$. Observe if $F(x) = x - \frac{1}{3}x^3$ then $F'(x) = 1 - x^2$. Thus, the fundamental theorem gives

$$\begin{aligned}\int_{-1}^{1}(1 - x^2)dx &= F(1) - F(-1)\\ &= 4/3.\end{aligned}$$

4. (10 points) The number of mosquitos in my backyard starts with 1 billion biting little monsters and increases at a rate of $n'(t)$ mosquitos per week. What does $1,000,000,000 + \int_0^{52} n'(t)dt$ represent?

The term $1,000,000,000$ in the sum represents the initial number of mosquitos. The fundamental theorem tells us that $\int_0^{52} n'(t)dt$ represents the change in number of mosquitos between the current time and 52 weeks from now. Thus, the expression gives how many mosquitos will be in the backyard in 52 weeks.

5. (10 points) Oil leaked from a tank at a rate of $r(t)$ gallons per hour. The rate decreased as time passed and values of the rate at two hour time intervals are given in the table below. Find upper and lower estimates for the total amount of oil that leaked out.

t	0	2	4	6	8	10
r(t)	10.2	8.4	7.1	6.2	5.7	5.3

It is given that this is a decreasing function so the left hand sum will be an overestimate and the right hand sum will be an underestimate. We have

$$\begin{aligned} \text{overestimate} &= (10.2 + 8.4 + 7.1 + 6.2 + 5.7) \cdot 2 \\ &= 75.2 \text{ gallons.} \end{aligned}$$

and

$$\begin{aligned} \text{underestimate} &= (8.4 + 7.1 + 6.2 + 5.7 + 5.3) \cdot 2 \\ &= 65.4 \text{ gallons.} \end{aligned}$$

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6. (10 points) Recall that if a particle travels at a constant velocity v meters/second, then in time T the particle has a displacement of $D = v \cdot T$. Suppose now that a particle travels at a possibly non-constant velocity $v(t)$ meters/second at time t. Use the method of slicing that was discussed in class to slice up the time interval $[a, b]$ to show that the displacement of the particle from time $t = a$ to $t = b$ is given by $\int_a^b v(t)dt$. Be sure to include all the relevant steps and explain your arguments!

Consider a short interval $[t, t+\Delta t]$ contained in the interval $[a, b]$. As long as Δt is very small, we can approximate the velocity to be constant on the interval $[t, t + \Delta t]$ so the displacement over the interval $[t, t+\Delta t]$ is approximately $v(t)\Delta t$. This is true for any such small interval. With this in mind, we break up the interval $[a, b]$ into n different subintervals of equal width $\Delta t = \frac{b-a}{n}$, in other words, we consider $a = t_0 < t_1 < \cdots < t_n = b$ with $t_{j+1} - t_j = \Delta t$ for $j = 0, \ldots, n - 1$. The total displacement, D, can be approximated by adding up the approximate displacements for each subinterval. We obtain

$$D \approx \sum_{j=0}^{n-1} v(t_j)\Delta t.$$

To obtain a better and better approximation we take more and more intervals. Thus, to get the exact value we consider the limit as n goes to infinity:

$$D = \lim_{n\to\infty} \sum_{j=0}^{n-1} v(t_j)\Delta t.$$

However, the right hand side is exactly how we defined the definite integral so we obtain

$$D = \int_a^b v(t)dt.$$

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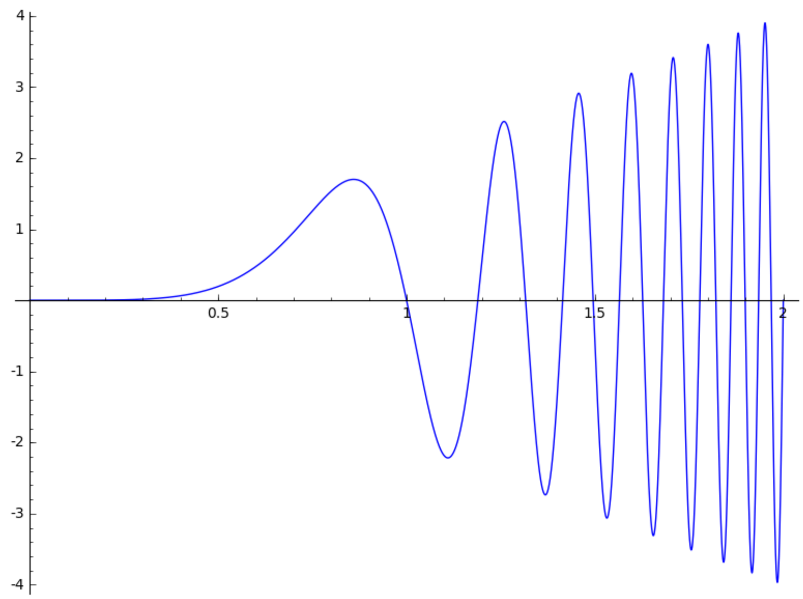
7. (5 points each) Let $F(x) = \int_0^{x^2} \sin(\pi t^2)dt$.

(a) Calculate $F'(x)$. Use your graphing calculator to give a graph of $F'(x)$ on the interval $[0, 2]$.

The second fundamental theorem of calculus and the chain rule gives

$$F'(x) = 2x\sin(\pi x^4).$$

The graph of the function (note: I used the wrong function here; it was supposed to be less maxs and mins):



(b) At what values of x in the interval $[0, 2]$ does $F(x)$ have local maximum values? Be sure to use calculus and not just your calculator to justify this.

The function $F(x)$ has extrema when $F(x) = 0$. These can be found using your calculator to be $x \approx 0, 1.19, 1.32, 1.41, 1.495, 1.57, 1.68, 1.73, 1.78, 1.82, 1.86, 1.899, 1.93, 1.97, 2$. The local maxs occur when $F'(x)$ goes from positive to negative as this corresponds to $F(x)$ going from increasing to decreasing. These occur at $1, 1.32, 1.495, 1.68, 1.78, 1.899, 1.97$.

(c) On what subintervals of $[0, 2]$ is $F(x)$ concave up? Be sure to justify this with calculus.

The function $F(x)$ is concave up when $F''(x) > 0$, which is when $F'(x)$ is increasing. To find these intervals, we use the calculator to find where the local mins and maxs occur for the function $F'(x)$ and then use the graph to see where $F'(x)$ is increasing. The intervals where $F(x)$ is concave up are $[0, 0.86]$, $[1.11, 1.26]$, $[1.37, 1.47]$, $[1.531, 1.597]$, $[1.66, 1.71]$, $[1.76, 1.80]$, $[1.84, 1.88]$, $[1.92, 1.95]$, $[1.98, 2]$.