Math 850 Final Homework

Name: ____

1. Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

2. A short exact sequence of R-modules

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is said to be *split* if $M_2 \cong M_1 \oplus M_3$ as R-modules. Show that the short exact sequence is split if and only if the associated sequence

$$0 \longrightarrow \operatorname{Hom}_R(M_3, N) \longrightarrow \operatorname{Hom}_R(M_2, N) \longrightarrow \operatorname{Hom}_R(M_1, N) \longrightarrow 0$$

is a short exact sequence of R-modules for each R-module N.

3. Let M be a Noetherian R-module and let $f : M \to M$ be a surjective R-module homomorphism. Show that $\ker(f^n) \cap \operatorname{image}(f^n) = 0$ for sufficiently large n.

4. Let R be an integral domain with fraction field K. Prove that every finitely generated R-submodule of K is a fractional ideal of R. If R is Noetherian, prove that $\mathfrak{a}$ is a fractional ideal of R if and only if $\mathfrak{a}$ is a finitely generated R-submodule of K.

5. Compute the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{10})$. Show that the integral closure is a Dedekind domain, but not a principal ideal domain.

6. (a) Let R be a ring and M a flat R-module. Let N_1 and N_2 be submodules of an R module N. Prove that as submodules of $N \otimes_R M$ one has

$$(N_1 \cap N_2) \otimes_R M \cong (N_1 \otimes_R M) \cap (N_2 \otimes_R M).$$

(b) Let $R \to S$ be a flat ring homomorphism. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be ideals of R. Prove that

$$(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e = \mathfrak{a}_1^e \cap \mathfrak{a}_2^e.$$

(c) If in addition one has $\mathfrak{a}_2$ is finitely generated, then

$$(\mathfrak{a}_1 : \mathfrak{a}_2)^e = (\mathfrak{a}_1^e : \mathfrak{a}_2^e).$$