# Math 581 Problem Set 8

Note that I may add another problem or two, but this is enough to get you started for next week.

**1.** Prove that a group $G$ is abelian if and only if the function $\varphi : G \to G$ given by $\varphi(g) = g^{-1}$ is a homomorphism of groups. In this case, show that $\varphi$ is an isomorphism.

**2.** Let $\varphi : G \to H$ be a homomorphism of groups.
**(a)** Let $G_1$ be a subgroup of $G$. Prove that $\varphi(G_1)$ is a subgroup of $H$. In particular, this shows that $\varphi(G)$ is a subgroup of $H$.
**(b)** Let $H_1$ be a subgroup of $H$. Prove that the set $\varphi^{-1}(H_1) = \{g \in G : \varphi(g) \in H_1\}$ is a subgroup of $G$. In particular, this shows that $\varphi^{-1}(H)$ is a subgroup of $G$.

**3.** **(a)** Prove that $H = \left\{ \begin{pmatrix} 1-n & -n \\ n & 1+n \end{pmatrix} \mid n \in \mathbb{Z} \right\}$ is a group under matrix multiplication.
**(b)** Prove that $H \cong \mathbb{Z}$.

**4.** List all the distinct left cosets of $H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$ in $S_3$. What is $[S_3 : H]$? Is $H$ a normal subgroup of $S_3$?

**5.** This is a series of finite group questions. They are not necessarily related.
**(a)** A group has fewer then 100 elements and subgroups of orders 10 and 25. What is the order of $G$?
**(b)** If $H$ and $K$ are subgroups of a finite group $G$, prove that $|H \cap K|$ is a common divisor of $H$ and $K$.
**(c)** If $G$ is a group with more then 1 element and $G$ has no proper subgroups, prove that $G$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime $p$.
**(d)** If $p$ and $q$ are primes, show that every proper subgroup of a group of order $pq$ is cyclic.

**6.** Let $a \in G$ be fixed, and define $\varphi : G \to G$ by $\varphi(x) = axa^{-1}$. Prove that $\varphi$ is a homomorphism. Under what conditions is $\varphi$ an isomorphism?

**7.** Show that $\varphi : \mathbb{R} \to \mathbb{C}^\times$, $\varphi(t) = \cos(2\pi t) + i\sin(2\pi t)$ is a homomorphism. What are its kernel and image?

**8.** Consider the subgroup $N = \mathbb{Z}$ of the additive group $\mathbb{Q}$.
**(a)** What does it mean for $r \equiv s (\text{mod}\, N)$, i.e., for the cosets $r + N = s + N$?
**(b)** Show that for $m, n \in \mathbb{Z}$, one has $m + N = n + N$.
**(c)** $\frac{1}{2} + N = ?$
**(d)** How many distinct cosets are there?

**9.** Consider the additive group $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Set $N = \langle (1, 2) \rangle$.
**(a)** What is $|N|$? What about $[G : N]$?
**(b)** Compute the distinct left cosets that comprise $G/N$.
**(c)** Prove that $N$ is a normal subgroup of $G$.
**(d)** Write out an addition table for $G/N$.
**(e)** What familiar group is $G/N$?