Math 581 Problem Set 6

1. Let $F \subseteq K$ be a finite field extension. Prove that if $[K : F] = 1$, then $K = F$.

2. Recall we showed that an angle θ is constructible if and only if $\cos\theta$ and $\sin\theta$ are both constructible.
(a) Show that if angles θ_1 and θ_2 are constructible, then so are angles $\theta_1+\theta_2$ and $\theta_1 - \theta_2$.
(b) Prove that if the regular mn-gon is constructible, i.e., one can construct an angle of $\frac{2\pi}{mn}$, then the regular m- and n-gons are constructible as well.
(c) Prove that if $\gcd(m, n) = 1$ and the regular m- and n-gons are both constructible, then the regular mn-gon is constructible.
(d) Show it is possible to trisect the angle $\frac{2\pi}{5}$ and construct a regular 15-gon.

3. Recall deMoivre's theorem from section 2.3: For any integer n one has $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
(a) Use deMoivre's theorem to find a formula for $\sin 7\theta$ that does not contain any $\cos\theta$'s.
(b) Plug in $\theta = \frac{2\pi}{7}$ to find a polynomial in $\mathbb{Z}[x]$ that has $\sin\left(\frac{2\pi}{7}\right)$ as a root.
(c) Prove that the polynomial you found in part (b) is irreducible in $\mathbb{Z}[x]$.
(d) Prove that the regular heptagon (7-gon) is not constructible.

4. **(a)** Show that $x^4 + x + 1$ is irreducible in $(\mathbb{Z}/2\mathbb{Z})[x]$.
(b) Use part (a) to construct a finite field $\mathbb{F}_{2^4}$ of order 16.
(c) Draw a diagram that shows all the subfields of $\mathbb{F}_{2^4}$.

5. Let F be a field of characteristic p.
(a) Prove that for every positive integer n, one has

$$(a + b)^{p^n} = a^{p^n} + b^{p^n}$$

for all $a, b \in F$. (Hint: use induction on n.)
(b) Now assume that in addition F is finite. Prove that the map $\phi : F \to F$ given by $\phi(a) = a^p$ is an isomorphism. Use this to conclude that every element of F has a p^{th} root in F.
(c) Let K be a finite field of characteristic p with $F \subset K$ and m a positive integer. Set $L = \{a \in K : a^{p^m} \in F\}$. Prove that L is a subfield of K that contains F.
(d) Prove $L = F$. (Hint: Think vector spaces. If $\{v_1, \dots, v_n\}$ is a basis of L over F, use parts (a) and (b) to prove that $\{v_1^{p^m}, \dots, v_n^{p^m}\}$ is linearly

independent over F, which implies $n = 1$.)

6. Write a critique of the "proof" given in the following article. Please use only material the author provides in the article to critique the proof.

http://www.washingtonpost.com/wp-dyn/content/blog/2006/02/15/BL2006021501989.html