Math 581 Problem Set 5

1. Show that the set $\{\sqrt{2}, \sqrt{2}+i, \sqrt{3}-i\}$ is linearly independent over $\mathbb{Q}$.

2. Let $F \subseteq K$ be fields with $[K : F] = p$ for some prime number p.
(a) Show that there is no field E so that $F \subsetneq E \subsetneq K$.
(b) Use part (a) to conclude there is no field F so that $\mathbb{R} \subsetneq F \subsetneq \mathbb{C}$.
(c) Let $\alpha \in K$ with $\alpha \notin F$. Prove that $K = F[\alpha]$.
(d) Use part (c) to conclude that $\mathbb{C} = \mathbb{R}[i]$.

3. Let V be a vector space over $\mathbb{Q}$. Prove that if $v, w \in V$ are linearly independent, then so are $v + w$, $2v - w$.

4. Prove that $\{v_1, \ldots, v_k\}$ is a basis for V if and only if every vector in V can be written uniquely as a linear combination of $v_1, \ldots, v_k$.

5. Give a basis and the degree of the field extension in each of the following cases:
(a) $V = \mathbb{Q}[\omega_7]$ over $\mathbb{Q}$ where ω_7 is a seventh root of unity
(b) $V = \mathbb{Q}[\omega_6]$ over $\mathbb{Q}[i]$ where ω_6 is a sixth root of unity
(c) $V = \mathbb{C}$ over $\mathbb{R}$
(d) $V = (\mathbb{Z}/7\mathbb{Z})[x]/\langle x^3 - 3\rangle$ over $\mathbb{Z}/7\mathbb{Z}$.

6. Let $f(x) = 2x^{15} - 49x^{12} + 21x^7 + 70x^2 + 35$. Let K be an extension field of $\mathbb{Q}$ with $[K : \mathbb{Q}] = 32$. Show K does not contain any roots of $f(x)$.

7. Let p be a prime number. Show that $\mathbb{Q}[\sqrt[21]{p}] = \mathbb{Q}[\sqrt[3]{p}, \sqrt[7]{p}]$.

8. Let p be a prime number.
(a) Let $n \in \mathbb{N}$. Show that $f(x) = x^n - p$ is irreducible.
(b) What is the degree of the field $\mathbb{Q}[\sqrt[n]{p}]$ over $\mathbb{Q}$?
(c) Use part (b) to show that $\mathbb{R}$ is not a finite extension of $\mathbb{Q}$.

9. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree n and let K be the splitting field of $f(x)$. Prove that $[K : \mathbb{Q}] \leq n!$.