Math 581 Problem Set 4

1. Find the greatest common divisor of $2+3i$ and $6-7i$ in $\mathbb{Z}[i]$. Write the greatest common divisor as a linear combination of $2+3i$ and $6-7i$.

An integral domain R is a *Euclidean domain* if there is a function δ from the nonzero elements of R to the nonnegative integers with these properties:
(i) If a and b are nonzero elements of R, then $\delta(a) \leq \delta(ab)$.
(ii) If $a, b \in R$ and $b \neq 0_R$, then there exist $q, r \in R$ such that $a = bq + r$ and either $r = 0_R$ or $\delta(r) < \delta(b)$.

2. Let p be an irreducible element in a Euclidean domain R. Prove that if $p|bc$, then $p|b$ or $p|c$.

3. Prove that every Euclidean domain is a PID.

An ideal $\wp$ in a commutative ring R is said to be *prime* if $\wp \neq R$ and whenever $bc \in \wp$, then $b \in \wp$ or $c \in \wp$. An ideal $\mathfrak{m}$ in a ring R is said to be *maximal* if $\mathfrak{m} \neq R$ and whenever I is an ideal such that $\mathfrak{m} \subset I \subset R$, then $\mathfrak{m} = I$ or $I = R$.

4. Prove that $\mathfrak{m}$ is a maximal ideal if and only if $R/\mathfrak{m}$ is a field.

5. Prove that every maximal ideal is a prime ideal.

6. List all the maximal ideals in $\mathbb{Z}/10\mathbb{Z}$.

7. Show that the principal ideal $\langle x-1 \rangle$ in $\mathbb{Z}[x]$ is a prime ideal but not a maximal ideal.

8. Let $\phi : R \to S$ be a surjective homomorphism of commutative rings. If $\wp$ is a prime ideal in S, prove that $\phi^{-1}(\wp)$ is a prime ideal in R. (Note this property is NOT true for maximal ideals!)

9. Suppose that $I \subsetneq R$ is an ideal with the property that every element $a \notin I$ is a unit. Prove that I is a maximal ideal.

10. Prove that the maximal ideals of $\mathbb{C}[x]$ are in a one-to-one correspondence with points of $\mathbb{C}$, i.e., there is a bijection between the set of maximal ideals in $\mathbb{C}[x]$ and $\mathbb{C}$.