Math 581 Problem Set 3

1. Prove that complex conjugation is a isomorphism from $\mathbb{C}$ to $\mathbb{C}$.

2. Let $a, b \in R$ and suppose $\langle a \rangle = \langle b \rangle$. What can we conclude about a and b?

3. Find all ideals in the ring $\mathbb{Z}/12\mathbb{Z}$.

4. Prove that the map $\phi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ defined by $\phi(x) = x^p$ is a ring homomorphism for p a prime. Find $\ker \phi$.

5. Use the ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for an appropriate value of m to prove that the equation $x^2 - 5y^2 = 2$ has no solution for $x, y \in \mathbb{Z}$.

6. Let R and S be commutative rings, and let $\phi : R \to S$ be a ring homomorphism.
(a) Give an ideal $J \subset S$, define

$$\phi^{-1}(J) = \{r \in R : \phi(r) \in J\} \subset R.$$

Prove that $\phi^{-1}(J)$ is an ideal.
(b) Given an ideal $I \subset R$, prove that

$$\phi(I) = \{\phi(r) : r \in I\} \subset S$$

is an ideal.
(c) Prove that every ideal in $\mathbb{Z}/m\mathbb{Z}$ is principal.

7. If $\gcd(m, n) = 1$ in $\mathbb{Z}$, prove that $\langle m \rangle \cap \langle n \rangle$ is the ideal $\langle mn \rangle$.

8. Let $\phi : R \to S$ be an isomorphism. Prove that:
(a) $\phi(a)$ is a unit if and only if a is a unit
(b) $\phi(b)$ is a zero-divisor if and only if b is a zero-divisor

9. Using the first isomorphism theorem, prove that $\mathbb{Q}[x]/\langle x^2+x+1 \rangle \cong \mathbb{Q}[\omega]$ where ω is a third root of unity.

10. Is $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/9\mathbb{Z}$? Be sure to justify your answer.

11. Let p be a prime number.
(a) Prove that $\mathbb{Q}[\sqrt{p}] \cong \mathbb{Q}[x]/\langle x^2 - p \rangle$.

(b) Prove that

$$\mathbb{Q}[\sqrt{p}] \cong \left\{ \begin{pmatrix} a & pb \\ b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}.$$