Math 581 Problem Set 2

1. Determine if the following polynomials are irreducible over $\mathbb{Q}$. If they are, prove it. If not, write them as a product of irreducible polynomials.
(a) $f(x) = 5x^{11} - 6x^4 + 12x^3 + 36x - 6$
(b) $f(x) = 2x^4 + 7x^3 + 5x^2 + 7x + 3$
(c) $f(x) = 9x^4 + 4x^3 - 3x + 7$

2. Let F be a field and $f(x) \in F[x]$. If $c \in F$ and $f(x+c)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$.

3. Let p be a prime. Prove that $\sqrt[n]{p} \notin \mathbb{Q}$ for all integers $n \geq 2$. (It may help to look at the polynomial $f(x) = x^n - p$.)

4. Show that there are infinitely many integers n such that $f(x) = x^9 + 12x^5 - 21x + n$ is irreducible in $\mathbb{Q}[x]$.

5. Prove that $f(x) \in (\mathbb{Z}/2\mathbb{Z})[x]$ has $x + \overline{1}$ as a factor if and only if it has an even number of nonzero coefficients.

6. Prove that for any prime p, $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$.

7. In this problem you will show that the polynomial $f(x) = x^4 - 10x^2 + 1$ is irreducible in $\mathbb{Q}[x]$ but is reducible in $(\mathbb{Z}/p\mathbb{Z})[x]$ for every prime p.
(a) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.
(b) Prove that $f(x)$ is reducible in $(\mathbb{Z}/p\mathbb{Z})[x]$ for every prime p. It may be helpful to use the method of undetermined coefficients along with the following lemma, which you may use without proof:
Lemma: If neither 2 nor 3 is a square modulo p, then 6 is a square modulo p.