Math 581 Problem Set 1

1. Find the splitting field of $f(x) = x^4 - 4x^2 - 5$ over $\mathbb{Q}$.

2. Let X be a finite set and $f : X \to X$ a function. Prove that f is injective if and only if f is surjective.

3. Let B be a set of n elements. Prove that the number of different injective functions from B to B is $n!$. (Hint: induction may be helpful here!)

4. Let $a, b, c \in \mathbb{Z}$. If $a|(b + c)$ and $gcd(b, c) = 1$, prove that $\gcd(a, b) = 1 = \gcd(a, c)$.

5. If $5|(a^2 + b^2 + c^2)$, prove $5|a$ or $5|b$ or $5|c$.

6. Let $a, n \in \mathbb{Z}$ with $n > 1$. Prove that $\gcd(a, n) = 1$ in $\mathbb{Z}$ if and only if the equation $\overline{a}x = \overline{1}$ in $\mathbb{Z}/n\mathbb{Z}$ has a solution.

7. Define a new addition and multiplication on $\mathbb{Q}$ by

$$r \oplus s = r + s + 1$$

and

$$r \odot s = rs + r + s.$$

Prove that with these new operations $\mathbb{Q}$ is a commutative ring. Is it an integral domain? (Note, this has new operations so is NOT a subring of anything you know of, so you have to check ALL the properties of being a ring.)

8. Let $f(x), g(x), h(x) \in F[x]$, with $f(x)$ and $g(x)$ relatively prime. If $f(x)|h(x)$ and $g(x)|h(x)$, prove that $f(x)g(x)|h(x)$.

9. Show that $x - 1_F$ divides $a_nx^n + \cdots + a_1x + a_0$ in $F[x]$ if and only if $a_0 + a_1 + \cdots + a_n = 0_F$.