MATH 581 — FIRST MIDTERM EXAM

April 21, 2006

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 10 problems.
3. Do not separate the pages of the exam.
4. Your proofs should be neat and legible. You may and should use the back of pages for scrap work.
5. If you are unsure whether you can quote a result from class or the book, please ask.
6. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	12	
2	8	
3	10	
4	8	
5	8	
6	10	
7	8	
8	8	
9	13	
10	15	
TOTAL	100	

1. (3 points each) Throughout this problem R is a commutative ring and F is a field.

(a) Define what it means for $I \subset R$ to be an ideal.

(b) Define what it means for $f(x) \in F[x]$ to be irreducible.

(c) Give an example of a reducible polynomial $f(x) \in \mathbb{Q}[x]$ that has no roots in $\mathbb{Q}$.

(e) Let $a \in R$. Define $\langle a \rangle$.

2. (3+5 points) Let $\phi : R \to S$ be a ring homomorphism between two commutative rings.

(a) Define $\ker\phi$.

(b) Prove that $\ker\phi$ is an ideal of R.

3. (5 points each) **(a)** Prove that if $I \subset R$ is an ideal and I contains a unit, then $I = R$.

(b) Let F be a field. Prove that the only ideals in F are $\langle 0_F \rangle$ and F.

4. (8 points) Let I and J be ideals in R. Prove that $I \cap J$ is an ideal in R.

5. (8 points) Prove that the map $\phi : \mathbb{Q}[\sqrt{7}] \to \mathbb{Q}[\sqrt{7}]$ given by $\phi(a + b\sqrt{7}) = a - b\sqrt{7}$ is an isomorphism.

6. (10 points) Prove that the composition of two isomorphisms is again an isomorphism. (You may NOT quote any results about composition of functions from Math 580!)

7. (8 points) Prove that $\mathbb{R}[x]/\langle x^2 + 1\rangle \cong \mathbb{C}$.

8. (8 points) Prove that $\sqrt[5]{\frac{17}{25}}$ is not a rational number. (It may be helpful to consider the polynomial $f(x) = 25x^5 - 17$.)

9. (3+5+5 points) Let $\phi : R \to S$ be a homomorphism between commutative rings.

(a) Define the image of ϕ, i.e., $\phi(R)$.

(b) Prove that $\phi(R)$ is a subring of S.

(d) Prove that if R is a field, then $\phi(R)$ is a field. (Problem 2(b) may be helpful here!)

10. (5 points each) An ideal $\wp$ in a commutative ring R is said to be a *prime ideal* if $\wp \neq R$ and whenever $ab \in \wp$, then $a \in \wp$ or $b \in \wp$.

(a) Let $p \in \mathbb{Z}$ be a prime number. Show that $\langle p \rangle$ is a prime ideal.

(b) Show that if $\wp$ is a prime ideal, then $R/\wp$ is an integral domain.

(c) Show that if $R/\wp$ is an integral domain, then $\wp$ is a prime ideal.