Math 580 Team Homework 4

Due: 2/24/06

1. We say a complex number z is a primitive n^{th} root of unity if $z^n = 1$ but $z^m \neq 1$ for all $0 < m < n$.
(a) Show that the number $\zeta_n = e^{\frac{2\pi i}{n}}$ is always a primitive n^{th} root of unity.
(b) Show that ζ_n^k is a primitive n^{th} root of unity if and only if $\gcd(k, n) = 1$.
(c) Show that if z is any primitive n^{th} root of unity, then $1, z, z^2, \ldots, z^{n-1}$ are distinct and comprise all the n^{th} roots of unity.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an isometry.
(a) Show that for any $z, w \in \mathbb{C}$, f carries the line joining z and w to the line joining $f(z)$ and $f(w)$.
(b) Prove that f preserves the (unsigned) measure of angles.

3. The altitude of a right circular cone is 6 and the radius of its base is 4. A right circular cylinder is inscribed in this cone and has volume 4/9 that of the cone. Find the altitude of the cylinder.

4. Let $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$ be the n^{th} roots of unity.
(a) Show that the conjugate of any n^{th} root of unity is another n^{th} root of unity. In particular, express $\bar{\zeta}^j$ in the form ζ^k for the appropriate k.
(b) Find the product of hte n^{th} roots of unity; this is written as $\prod_{k=0}^{n-1} \zeta^k$.
(c) Find the sum $\sum_{k=0}^{n-1} \zeta^k$ of the n^{th} roots of unity.

5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an isometry and $|f(z) - z| \leq N$ for some fixed $N \in \mathbb{R}_{>0}$ for all $z \in \mathbb{C}$. Prove that f must be a translation or the identity map.