Math 580 Team Homework 3

Due: 2/10/06

1. Let R be a ring. An element $a \in R$ is **nilpotent** if $a^n = 0_R$ for some positive integer n.
(a) Let a and b be nilpotent elements in a commutative ring R. Prove that $a + b$ and ab are also nilpotent.
(b) Let $\mathfrak{N}$ be the set of nilpotent elements of R. Show that $\mathfrak{N}$ is a subring of R.

2. Let $K \subset \mathbb{R}$. Show that $\mathbb{Q} \subset K$.

3. Prove that $\frac{1}{2} \notin \mathbb{Z}$. (Hint: See section 1.1)

4. Let R be an integral domain having a finite number of elements. Prove R is a field.

5. Prove or disprove: Between every two distinct rational numbers there are infinitely many rational numbers.

6. Prove that $\mathbb{Q}$ is a countable set, i.e., can be put in a one-to-one correspondence with $\mathbb{N}$. (Hint: Arrange the positive fractions $\frac{m}{n}$ in a two-dimensional array - numerators increasing along rows, denominators increasing down columns. Delete duplicates, and then count along 45° lines.)