Math 580 Team Homework 1

Due: 1/20/06

1. Suppose n is an odd integer. Prove:
(a) The equation $x^2 + x - n = 0$ has no solution $x \in \mathbb{Z}$. (Hint: Let D be the discriminant of this quadratic equation. For the equation to have an integer solution D must be a perfect square. Consider dividing the discriminant by 8 and looking at the remainder. Show this cannot be a perfect square and explain how this gives the desired result.)
(b) Prove that for any $m \in \mathbb{Z}$, the equation $x^2 + 2mx + 2n = 0$ has no solution $x \in \mathbb{Z}$.

2. **(a)** Let $f : B \to C$ and $g : C \to D$ be functions such that $g \circ f$ is surjective. Prove that g is surjective.
(b) Given an example of the situation in part (a) in which f is not surjective.

3. If n women play in a round-robin tennis tournament (each woman playing each other woman once), is it always possible to arrange their names in a vertical list so that each player on the list has beaten the other person just beneath her on the list?

4. Let $n \in \mathbb{N}$, and suppose you choose $n + 1$ distinct numbers from the list $1, 2, 3, \ldots, 2n - 1, 2n$. Will it always be the case that among the numbers you've chosen, you can find a pair with the property that one divides the other? Prove or provide a counter-example.

5. Let n be a positive integer. Suppose that there are three pegs and on one of them n rings are stacked, with each ring being smaller in diameter than the one below it. The game is to transfer all the rings to another peg according to these rules: (i) only one ring may be moved at a time; (ii) a ring may be moved to any peg but may never be placed on top of a smaller ring; (iii) the final order of the rings on the new peg must be the same as their original order on the first peg. Prove that the game can be completed in $2^n - 1$ moves and cannot be completed in fewer moves.