Supplemental Problems for 3.2

Jim Brown

1. Consider the ring $R = \mathbb{Q}[x]/(x^3 - x + 1)$.

(a) Find a polynomial $h(x)$ of degree less then 3 so that $g(x) = x^5 + 4x^2 + 10$ is congruent to $h(x)$.

(b) Finish the following statement: "$\overline{x^7} =$ ___ in R".

(c) Is this ring a field? Justify your answer.

2. Recall that the n^{th} roots of unity are the roots of the polynomial $\phi_n(x) = x^n - 1$. Note that $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$.

(a) Show that if we set $\omega = e^{\frac{2\pi i}{n}}$, then the roots of $x^{n-1} + x^{n-2} + \cdots + x + 1$ are given by $\omega, \omega^2, \cdots, \omega^{n-1}$.

(b) Let p be a prime. Is the ring $\mathbb{Q}[x]/(x^p - 1)$ a field? Is the ring $\mathbb{Q}[x]/(x^{p-1} + x^{p-2} + \cdots + x + 1)$ a field? Justify your answers!

(c) Consider the ring $\mathbb{Q}[x]/(x^{p-1} + x^{p-2} + \cdots + x + 1)$. Show that there is an element (not equal to 1) so that when you raise it to the p^{th} power you get 1.

3. Denote the field obtained by adjoining the third root of unity $\omega = e^{\frac{2\pi i}{3}}$ to $\mathbb{Q}$ by $\mathbb{Q}[\omega]$. This field is given by

$$\mathbb{Q}[\omega] = \left\{a + b\omega + c\omega^2 | a, b, c \in \mathbb{Q}\right\}.$$

(a) Let $a + b\omega + c\omega^2$ and $d + e\omega + f\omega^2$ be elements in $\mathbb{Q}[\omega]$. Compute their sum and product and write it in a form so that it is clear that it is in $\mathbb{Q}[\omega]$.

(b) Determine a polynomial $f(x)$ so that $\mathbb{Q}[\omega]$ is isomorphic to $\mathbb{Q}[x]/(f(x))$ (remember your $f(x)$ must be such that $\mathbb{Q}[x]/(f(x))$ is actually a field!)

(c) Show that $\mathbb{Q}[\omega] = \mathbb{Q}[\sqrt{3}i]$ by showing containment in each direction.

4. Consider the ring $(\mathbb{Z}/5\mathbb{Z})[x]/(x^2 - 2)$.

(a) Show this is a field.

(b) List all the elements of this field.

(c) Compute $\overline{2x+3} + \overline{4x+1}$ and $\overline{2x+3} \cdot \overline{4x+1}$.

(d) Find a polynomial $r(x)$ of degree smaller then 2 so that $\overline{f(x)} = \overline{r(x)}$ where $f(x) = x^7 + 3x^2 + 8$.

5. Let p be a prime and let $f(x)$ be an irreducible polynomial of degree n. How many elements are there in the field $(\mathbb{Z}/p\mathbb{Z})[x]/(f(x))$?