Sample homework solutions for 2.3

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2. Recall that the conjugate of the complex number $z = a + bi$ is defined to be $\overline{z} = a - bi$. Prove the following properties of the conjugate:

(c) $\overline{z} = z \Longleftrightarrow z \in \mathbb{R}$ and $\overline{z} = -z \Longleftrightarrow iz \in \mathbb{R}$

Proof: Observe that $\overline{z} = z$ if and only if $b = 0$ which is true if and only if $z \in \mathbb{R}$. This proves the first claim. For the second, observe that $\overline{z} = -z$ if and only if $a = 0$ which is true if and only if $z = bi$. This is equivalent to the statement that $iz \in \mathbb{R}$. ■

6. Use Corollary 3.3 to express the following in terms of $\sin\theta$ and $\cos\theta$ (the binomial theorem may prove helpful):

(b) $\cos 3\theta$

Note that $\cos 3\theta$ is the real part of the expression $(\cos\theta + i\sin\theta)^3$ by Corollary 3.3. Therefore, we have (using the binomial theorem) that

$$\cos 3\theta = \cos^3\theta - \cos\theta\sin^2\theta.$$

11. Express the following n^{th} roots of unity in the form $a + bi$.

(a) $n = 8$

Note that the 8^{th} roots of unity are given by $1, \omega, \omega^2, \ldots, \omega^7$ where

$$\omega = e^{\frac{2\pi i}{8}} = \cos\left(\frac{2\pi i}{4}\right) + i\sin\left(\frac{2\pi i}{8}\right).$$

Thus, we have that the 8^{th} roots of unity are given by:

$$\begin{aligned}
1 &= 1+0i \\
\omega &= \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} \\
\omega^2 &= i \\
\omega^3 &= -\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} \\
\omega^4 &= -1 \\
\omega^5 &= -\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2} \\
\omega^6 &= -i \\
\omega^7 &= \frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}
\end{aligned}$$

19. Let $\mathbb{Q}\left[\sqrt{-5}\right] = \left\{a+b\sqrt{-5} : a,b \in \mathbb{Q}\right\} \subset \mathbb{C}$. Show that $\mathbb{Q}\left[\sqrt{-5}\right]$ is a field.

Proof: To ease notation set $K = \mathbb{Q}\left[\sqrt{-5}\right]$. Since $K \subset \mathbb{C}$ and $\mathbb{C}$ is a field, we need only show that K is a subring of $\mathbb{C}$ and that multiplicative inverses of elements in K also lie in K. Let $a+b\sqrt{-5}$ and $c+d\sqrt{-5}$ be in K.

closed under addition: $(a+b\sqrt{-5})+(c+d\sqrt{-5}) = (a+c)+(b+d)\sqrt{-5} \in K$ since $a+c, b+d \in \mathbb{Q}$.

closed under multiplication: $(a+b\sqrt{-5})(c+d\sqrt{-5}) = (ac-5bd)+(ad+bc)\sqrt{-5} \in K$ since $ac-5bd$ and $ad+bc$ are in $\mathbb{Q}$.

additive identity: $0 = 0+0\sqrt{-5} \in K$

multiplicative identity: $1 = 1+0\sqrt{-5} \in K$

additive inverse: $-a-b\sqrt{-5} \in K$ since $-a$ and $-b$ are in $\mathbb{Q}$.

Thus we have that K is a subring of $\mathbb{C}$. Now observe that if $a+b\sqrt{-5} \neq 0$, then

$$\begin{aligned}\frac{1}{a+b\sqrt{-5}} &= \frac{a-b\sqrt{-5}}{(a+b\sqrt{-5})(a-b\sqrt{-5})}\\ &= \frac{a-b\sqrt{-5}}{a^2+5b^2}\\ &= \left(\frac{a}{a^2+5b^2}\right)-\left(\frac{b}{a^2+5b^2}\right)\sqrt{-5}\in K\end{aligned}$$

since $\frac{a}{a^2+5b^2}$ and $-\frac{b}{a^2+5b^2}$ are both in $\mathbb{Q}$. Thus K is a field. ■