Sample homework solutions for 2.2

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3. Prove that for $x, y \in \mathbb{R}^+$, $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Proof: Note that this is equivalent to proving that $x + y \geq 2\sqrt{xy}$. Using problem 2 of this section (which is proved by induction on n), we see that $x + y \geq 2\sqrt{xy}$ if and only if $(x + y)^2 \geq (2\sqrt{xy})^2$ if and only if $x^2 + 2xy + y^2 \geq 4xy$ if and only if $x^2 + 2xy + y^2 - 4xy \geq 0$. Now observe that $x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x - y)^2$. Using that $(x - y)^2 \geq 0$ for all $x, y \in \mathbb{R}^+$, we have the statement. ■

5. Do the irrational numbers form a field? In particular, is it true that if a and b are irrational numbers, then $a + b$ and ab are necessarily irrational numbers?

The irrational numbers do not form a field. The easiest way to see this is to observe that $0 \in \mathbb{Q}$, so 0 is not an irrational number. Thus there is no 0 in the irrational numbers. We know that $\sqrt{2}$ is an irrational number, however, $\sqrt{2}\sqrt{2} = 2 \in \mathbb{Q}$, so the irrationals are not closed under multiplication. We also have that $-\sqrt{2}$ is an irrational number, but $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$, so the irrationals are not closed under addition either.

6. Prove that the following numbers are irrational:

(a) $\sqrt{3}$

Proof: We apply the exact same argument used to prove that $\sqrt{2}$ is irrational, this time reaching a contradiction that if $\sqrt{3} = \dfrac{a}{b}$ with $\gcd(a, b) = 1$, then we show that $3|a$ and $3|b$. ■

(b) $\sqrt[3]{2}$

Proof: Suppose there exists $\dfrac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$ and $\sqrt[3]{2} = \dfrac{a}{b}$. We can write $a^3 = 2b^3$. In particular we have that $2|a^3$. Now applying homework problem 9 from 1.2 we see that $2|a$. Thus there exists a $k \in \mathbb{Z}$ so that $a = 2k$. Plugging this into our original equation we have $b^3 = 4k^3$. This implies that $4|b^3$, in particular, $2|b^3$. As above, this implies that $2|b$. This is a contradiction as we assumed $\gcd(a,b)=1$. ■

(c) $\log_{10} 3$

Proof: Suppose there exists $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a,b) = 1$, a and b not both negative, and $\log_{10} 3 = \frac{a}{b}$. This translates to the statement that $10^{\frac{a}{b}} = 3$, i.e., $10^a = 3^b$. So we have reduced the problem to showing that this equation cannot occur. If $a \leq 0$, then by assumption $b > 0$ and $10^a \leq 1 < 3^b$, so we cannot have equality. Similarly if $b < 0$. So we are reduced to the case that a and b are both positive integers. Again there are several ways to show this equality cannot occur. The easiest probably being to observe that 10^a is an even integer where 3^b is an odd integer so we cannot have equality. ■

(d) $\sqrt{2}+\sqrt{3}$

Proof: Suppose there exists $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a,b) = 1$, a and b not both negative, and $\sqrt{2}+\sqrt{3} = \frac{a}{b}$. Squaring both sides we have $2+2\sqrt{6}+3 = \left(\frac{a}{b}\right)^2$. Solving this for $\sqrt{6}$ we see that $\sqrt{6} \in \mathbb{Q}$. So there exists $\frac{c}{d} \in \mathbb{Q}$ with $\gcd(c,d) = 1$ so that $\sqrt{6} = \frac{c}{d}$. As in part (a) we get that $c^2 = 6d^2$. Thus, $2 \mid c^2$, which implies $2 \mid c$. So we can write $c = 2k$ for some integer k. Our equation becomes $2k^2 = 3d^2$. This shows that $2|3d^2$, but since $2 \nmid 3$, we must have $2|d^2$ by Proposition 2.5 of Chapter 1. Thus $2|d$, a contradiction to the fact that $\gcd(c,d) = 1$. Thus it must be that $\sqrt{2}+\sqrt{3}$ is irrational. ■

7. Elaborate on the density principle enunciated in Proposition 1.4 as follows:

(a) Using the fact that $0 < \frac{\sqrt{2}}{2} < 1$, prove that between any two distinct rational numbers there is an irrational number.

Proof: We will show this statement in two different ways. First, observe that if $a \in \mathbb{Q}$ then $a+\sqrt{2}$ is an irrational number. For suppose it were rational, then we would have $a+\sqrt{2} = b \in \mathbb{Q}$, but this implies that $\sqrt{2} = b - a \in \mathbb{Q}$, which is a contradiction. Let $r < s$ be two rational numbers. Then we have $r+\sqrt{2} < s+\sqrt{2}$. Apply Proposition 2.4 which

states that between any two real numbers is a rational number to conclude that there exists a $c \in \mathbb{Q}$ so that $r + \sqrt{2} < c < s + \sqrt{2}$. Now we just subtract $\sqrt{2}$ to get $r < c + \sqrt{2} < s$ and from above we have that $c + \sqrt{2}$ is an irrational number.
For the second proof, observe that if $a \in \mathbb{Q}$ with $a \neq 0$, then $a\sqrt{2}$ is an irrational number. If it were rational, we would have $a\sqrt{2} = b$ for some $b \in \mathbb{Q}$. But then $\sqrt{2} = \dfrac{b}{a} \in \mathbb{Q}$, a contradiction. Now suppose we have two rational numbers $r < s$. Thus we have $\sqrt{2}r < \sqrt{2}s$. Using Proposition 2.4 again we see that there is a rational number c so that $\dfrac{r}{\sqrt{2}} < c < \dfrac{s}{\sqrt{2}}$, i.e., $r < \sqrt{2}c < s$ and we are done. ■

(b) Deduce that between any two distinct real numbers there is an irrational number.

Proof: Let $x < y$ be two distinct real numbers. By Proposition 2.4 there is a rational number r so that $x < r < y$. Now treating r as a real number, we apply Proposition 2.4 to $r < y$ to obtain a rational number s so that $r < s < y$. Thus we have $x < r < s < y$. Now apply part (a) to find an irrational number z so that $r < z < s$ and we are done. ■