Sample homework solutions for 2.1

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2. Prove Lemma 1.1.

Lemma 1.1 The following laws of inequalities are valid in any ordered field F:
(i) If $a > b$, then $a + c > b + c$ for all $c \in F^+$.
(ii) If $a > b$ and $c > 0$, then $ac > bc$.
(iii) If $a > b$ and $c < 0$, then $ac < bc$.
(iv) If $a > b$ and $b > c$, then $a > c$.
(v) If $a \neq 0$, then $a^2 > 0$.

Proof: We proved some of these in class. The only two that are not completely analogous to the ones proved in class are (i) and (v), so we prove those.
(i) Let $a > b$, i.e., $a - b \in F^+$. Observe that $(a + c) - (b + c) = a - b \in F^+$, so we are done.
(v) Let $a \neq 0$. By the definition of ordered field, we have that $a \in F^+$ or $-a \in F^+$. If $a \in F^+$, then $a \cdot a = a^2$ must be in F^+ as well since F^+ is closed under addition. Similiarly, we obtain that $(-a)(-a) = a^2 \in F^+$ where we have used exercise 6(c) from section 1.4 here. ■

3. Prove or give a counterexample. If the statement is false, modify it to make it correct.
(a) if $a > b$ and $c > d$, then $ac > bd$;

This statement is false. A simple counterexample would be $-1 > -2$ and $0 > -1$, but 0 is not larger then 2. However, if we add the condition that $b > 0$ and $d > 0$, then it becomes a true statement. To prove this, observe that $a > b$ and $c > d > 0$ implies by Lemma 1.1 (ii) that $ac > bc$. Now use that $c > d$ and $b > 0$ to conclude that $bc > bd$ using Lemma 1.1 (ii) again. Combining these two inequalities by Lemma 1.1 (iv) we have $ac > bd$. ■

5. Prove that for $r, s \in \mathbb{Q}$, each of the following inqualities holds. In each case, also determine when equality holds.
(a) $r^2 + s^2 \geq 2rs$

Proof: Observe that $r^2 + s^2 - 2rs = (r - s)^2$. Since $r - s \in \mathbb{Q}$, we see that $(r - s)^2 \geq 0$ for all $r, s \in \mathbb{Q}$. Rewriting this, $r^2 + s^2 \geq 2rs$. It is also easy to see now that equality holds if and only if $r = s$. ■

7. A common mistake made by high school algebra students (and, alas, by college calculus students alike) is the following:

$$\frac{a}{b} + \frac{a}{c} = \frac{a}{b+c}.$$

(a) Show that in any field this equation implies that either $a = 0$ or $b^2 + bc + c^2 = 0$.

Proof: Basically we just need to get common denominators and bring everything to one side:

$$\frac{a(b^2 + c^2 + bc)}{bc(b+c)} = 0.$$

Now we use that there are no zero-divisors in a field to conclude that $a = 0$ or $b^2 + c^2 + bc = 0$.

(b) Show that in $\mathbb{Q}$ it holds only when $a = 0$.

Proof: It clearly holds when $a = 0$, so we only need to show that $b^2 + c^2 + bc$ can never be zero for $b, c \in \mathbb{Z}$. Suppose that it can be zero, i.e., there are $b, c \in \mathbb{Z}$ so that $b^2 + bc + c^2 = 0$. Dividing both sides by c^2 we get that

$$\left(\frac{b}{c}\right)^2 + \left(\frac{b}{c}\right) + 1 = 0.$$

Making the substitution $x = \frac{b}{c}$, we see that this amounts to having a rational solution to the polynomial $x^2 + x + 1$, which using the quadratic equation we see has no rational roots.

(c) Give an example of a field where $a \neq 0$ and $b^2 + bc + c^2 = 0$ instead.

Proof: The field $\mathbb{Z}/3\mathbb{Z}$ is such a field. Just set $b = 1 = c$ and $a = 1$. Then $b^2 + bc + c^2 = \bar{3} = \bar{0}$ and $a \neq \bar{0}$. ■