Sample homework solutions for 1.3

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8. Determine the last digit of 3^{400}; then the last two digits. Determine the last digit of 7^{99}.

Note that the last digit of an integer n is precisely the number a so that $n \equiv a \pmod{10}$. Similarly, the last two digits is the number b so that $n \equiv b \pmod{100}$. Observe that $3^4 = 81 \equiv 1 \pmod{10}$. Thus, we have that

$$\begin{aligned} 3^{400} &= (3^4)^{100} \\ &\equiv 1^{100} \pmod{10} \\ &\equiv 1 \pmod{10}. \end{aligned}$$

Thus, the last digit of 3^{400} must be 1.
Observe now that $3^8 = 6561 \equiv 61 \pmod{100}$. We continue with this pattern:

$$\begin{aligned} 3^{16} &\equiv 61 \cdot 61 \pmod{100} \\ &\equiv 21 \pmod{100} \\ 3^{32} &\equiv 21 \cdot 21 \pmod{100} \\ &\equiv 41 \pmod{100} \\ 3^{64} &\equiv 81 \pmod{100} \\ 3^{128} &\equiv 61 \pmod{100} \\ 3^{256} &\equiv 21 \pmod{100}. \end{aligned}$$

Now observe that $400 = 256 + 128 + 16$, so we have

$$\begin{aligned} 3^{400} &= 3^{256} \cdot 3^{128} \cdot 3^{16} \\ &\equiv 21 \cdot 61 \cdot 61 \pmod{100} \\ &\equiv 41 \pmod{100}. \end{aligned}$$

Thus, the last two digits of 3^{400} are 4 and 1.
Observe that $7^4 = 2401 \equiv 1 \pmod{10}$. Writing $99 = 24(4) + 3$, we have

$$\begin{aligned} 7^{99} &= (7^4)^{24} 7^3 \\ &\equiv 7^3 \pmod{10} \\ &\equiv 3 \pmod{10}. \end{aligned}$$

Thus the last digit of 7^{99} is 3.

17. Suppose m and n are positive integers. Show that $3^m + 3^n + 1$ cannot be a perfect square.

Proof: Observe that any perfect square must be congruent to 0, 1, or 4 modulo 8. One sees this by checking all possible cases, i.e., looking at $0^2 \pmod 8), 1^2 \pmod 8), 2^2 \pmod 8), \ldots, 7^2 \pmod 8)$. Also note that $3^2 \equiv 1 \pmod 8$. Now we can write $m = 2q_1 + r_1$ and $n = 2q_2 + r_2$ for $0 \leq r_1, r_2 < 2$. We want to show that $3^m + 3^n + 1$ cannot be congruent to 0, 1, or 4 modulo 8, which will show it cannot be a perfect square. Note that

$$3^m = 3^{2q_1 + r_1} = (3^2)^{q_1} 3^{r_1} \equiv 3^{r_1} \pmod 8.$$

Similarly we have $3^n \equiv 3^{r_2} \pmod 8$. Now there are only a few cases to check as the only possibilities for r_1 and r_2 are 0, 1, and 2.
Case 1: If $r_1 = r_2 = 0$, then $3^m + 3^n + 1 \equiv 1 + 1 + 1 \equiv 3 \pmod 8$. Since 3 is not a possibility for a perfect square, this shows we cannot have a perfect square in this case.
Case 2: If $r_1 = 0, r_2 = 1$, then $3^m + 3^n + 1 \equiv 5 \pmod 8$, again an impossibility for a perfect square. Note this case also handles the case of $r_1 = 1$ and $r_2 = 0$ by symmetry of the equation.
Case 3: If $r_1 = r_2 = 1$, then $3^m + 3^n + 1 \equiv 7 \pmod 8$, again an impossibility for a perfect square.
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21. (c) Use the Chinese Remainder Theorem, Theorem 3.7, to solve the following simultaneous congruences:

$$\begin{aligned} x &\equiv 3 \pmod 4, \\ x &\equiv 4 \pmod 5, \\ x &\equiv 3 \pmod 7. \end{aligned}$$

We begin by solving the simultaneous congruences

$$\begin{aligned} x &\equiv 3 \pmod 4, \\ x &\equiv 4 \pmod 5. \end{aligned}$$

Observe that $1 = 5(1) + 4(-1)$. Therefore a solution to this set of congruences is given by $x = 3(5) + 4(-4) = -1$. We could either leave it in this form or convert it to a positive solution modulo 20. The positive solution

would then be $x = 19$.
Now we solve the simultaneous congruences

$$\begin{aligned} x &\equiv 19 \pmod{20}, \\ x &\equiv 3 \pmod{7}. \end{aligned}$$

Observe that $1 = 7(3) + 20(-1)$. Therefore a solution to this set of congruences is given by $x = 19(7(3)) + 3(-20) = 339$. The least common multiple of 20 and 7 is 140, so the solution reduces to $x \equiv 59 \pmod{140}$. Thus, the smallest positive solution to the original 3 congruences is $x = 59$.