Sample homework solutions for 1.1

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4. Prove the following by induction:

b. The sum of the first n odd integers is n^2.

Proof: We prove the statement by induction. The base case of $n = 1$ is clear as $1 = 1^2$. Suppose the statement is true for $n = k$, i.e.,

$$1 + 3 + 5 + \cdots + 2k - 1 = k^2.$$

We now consider the case of $n = k + 1$:

$$\begin{aligned} 1 + 3 + 5 + \cdots + 2k + 1 &= (1 + 3 + 5 + \cdots + 2k - 1) + 2k + 1 \\ &= k^2 + 2k + 1 \qquad \text{(by our induction hypothesis)} \\ &= (k+1)^2. \end{aligned}$$

Thus, we see the statement holds for $n = k + 1$. Therefore by induction the statement is true for all positive integers n. ■

d. For $n \geq 1$, $n^3 - n$ is divisible by 3.

Proof: We prove the statement by induction. The base case of $n = 1$ is clear as $1^3 - 1 = 0$, which is divisible by 3. Suppose the statement is true for $n = k$, i.e., there exists an integer m so that $3m = k^3 - k$. We now consider the case of $n = k + 1$. Observe that we have

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3(k^2 + k) \\ &= 3m + 3(k^2 + k) \qquad \text{(by our induction hypothesis)} \\ &= 3(m + k^2 + k). \end{aligned}$$

Thus, we see that $3|(k+1)^2 - (k+1)$. Therefore by induction the statement is true for all positive integers n. ■

h. $3^n \geq 2n + 1$ for all $n \in \mathbb{N}$.

Proof: We prove the statement by induction. The base case of $n = 1$ follows as we observe that $3^1 = 3 = 2(1) + 1$. Suppose that $3^k \geq 2k + 1$. Observe that we have

$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(2k+1) \qquad \text{(by our induction hypothesis)} \\ &= 6k+3 \\ &\geq 2(k+1)+1 \qquad \text{(since } 6k \geq 2k \text{ for } k \geq 1). \end{aligned}$$

Thus, we see that $3^{k+1} \geq 2(k+1)+1$. Hence, by induction the statement is true for all $n \in \mathbb{N}$. ■

6. **a.** Using the original definition of $\binom{n}{k}$, prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof: The original definition of $\binom{n}{k}$ was that it is the number of ways to choose a k element subset from a set of n elements. We can split our set of subsets into the subsets that contain n and the subsets that do not contain n. Tthe subsets that do not contain n are precisely all of the subsets of the $n-1$ element set $\{1, 2, \ldots, n-1\}$. Thus, there are $\binom{n-1}{k}$ possible subsets that do not contain n. If we desire a k element subset that must contain n, what we are really asking for it $k-1$ element subsets of the set $\{1, 2, \ldots, n-1\}$. There are precisely $\binom{n-1}{k-1}$ such sets. Thus, the total number of k element subsets of an n element set is $\binom{n-1}{k} + \binom{n-1}{k-1}$, as desired. ■

b. Prove by induction on n that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof: The statement is clear for the base case of $n = 1$ as we can easily check the equation for $k = 0, 1$. Now suppose that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Observe that we have

$$\begin{aligned} \binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \qquad \text{(by part a.)} \\ &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} \qquad \text{(by our induction hypothesis)} \\ &= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!} \\ &= \frac{(n+1)!}{k!(n+1-k)!}. \end{aligned}$$

Thus, by induction, the statement is true for all positive integers n. ■