Sample homework solutions for A.2

Jim Brown

These are solutions that are meant to not just show you how to do the problems, but how to write them up properly as well. They are **not** written to be as easy as possible for you to read. In particular, they do not contain things such as "I need to show *blank* to prove *blank* is true." Statements such as those would be helpful if you are having trouble with the material and you should see me outside of class if you are having trouble following these solutions.

2. Let X and Y be arbitrary sets. Let $X - Y = \{x : x \in X \text{ and } x \notin Y\}$. Prove that $(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$.

Proof: Let $x \in (X - Y) \cup (Y - X)$. By the definition of union one has that $x \in X - Y$ or $x \in Y - X$. If $x \in X - Y$, then $x \in X$ and $x \notin Y$. In particular, since $x \in X$, x must be in $X \cup Y$. Using that $x \notin Y$ shows that $x \notin X \cap Y$. Thus we see that $x \in (X \cup Y) - (X \cap Y)$. If $x \in Y - X$, the exact same argument shows that $x \in (X \cup Y) - (X \cap Y)$. Hence we have shown that $(X - Y) \cup (Y - X) \subseteq (X \cup Y) - (X \cap Y)$.
Let $x \in (X \cup Y) - (X \cap Y)$. Thus $x \in X \cup Y$, i.e., $x \in X$ or $x \in Y$. Suppose that $x \in X$. Necessarily $x \notin Y$ for otherwise $x \in X \cap Y$. Thus $x \in X - Y$. Similarly, if $x \in Y$ we get that $x \in Y - X$. Thus we have shown $(X \cup Y) - (X \cap Y) \subseteq (X - Y) \cup (Y - X)$. Since have shown containment in each direction, the sets must be equal as claimed. ■

3. Let $f : X \to Y$. Let $A, B \subset X$ and $C, D \subset Y$. Prove or give a counterexample (if possible, provide sufficient hypotheses for each statement to be valid):

(a) $f(A) \cup f(B) = f(A \cup B)$

Proof: Let $y \in f(A) \cup f(B)$. The definition of union gives that $y \in f(A)$ or $y \in f(B)$. If $y \in f(A)$, then there exists $a \in A$ so that $f(a) = y$. Since $A \subset A \cup B$, we have that $a \in A \cup B$ and $f(a) = y$, i.e., $y \in f(A \cup B)$. Similarly, if $y \in f(B)$ one gets $y \in f(A \cup B)$ by applying the same argument. Hence we have that $f(A) \cup f(B) \subseteq f(A \cup B)$.
Let $y \in f(A \cup B)$. There exists $x \in A \cup B$ so that $f(x) = y$. The fact that $x \in A \cup B$ implies that $x \in A$ or $x \in B$. If $x \in A$, then $y \in f(A)$. If $x \in B$,

then $y \in f(B)$. Thus we have $f(A \cup B) \subseteq f(A) \cup f(B)$. Since we have shown containment in each direction, the sets must be equal as claimed. ■
(e) $f^{-1}(D) \cap f^{-1}(D) = f^{-1}(C \cap D)$

Proof: Let $x \in f^{-1}(D) \cap f^{-1}(D)$, i.e., $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. Using the definition of $f^{-1}(C)$ shows that $f(x) \in C$. Similarly we get that $f(x) \in D$. Thus, $f(x) \in C \cap D$. In particular, $x \in f^{-1}(C \cap D)$. Thus we have that $f^{-1}(D) \cap f^{-1}(D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.
Let $x \in f^{-1}(C \cap D)$, i.e., $f(x) \in C \cap D$. Thus we have that $f(x) \in C$ and $f(x) \in D$. The fact that $f(x) \in C$ implies that $x \in f^{-1}(C)$ and the fact that $f(x) \in D$ implies that $x \in f^{-1}(D)$. Thus, $x \in f^{-1}(D) \cap f^{-1}(D)$ and so $f^{-1}(C \cap D) \subseteq f^{-1}(D) \cap f^{-1}(D)$. Since we have shown containment in each direction, the sets must be equal as claimed. ■

(g) $f(f^{-1}(C)) = C$

This statement is false. Consider the set $C = \{\pm 1\}$ and the function $f(x) = x^2$. The set $f^{-1}(C)$ is equal to $\{\pm 1\}$. However, applying f to this set we see that $f(f^{-1}(C)) = \{1\} \neq C$.

Claim: If the function f is surjective, then $f(f^{-1}(C)) = C$.

Proof: We know from the book already that $f(f^{-1}(C)) \subseteq C$ (see page 376). (However, here is a quick proof: Let $c \in f(f^{-1}(C))$. By definition, this means there exists $x \in f^{-1}(C)$ so that $f(x) = c$. The fact that $x \in f^{-1}(C)$ says that $f(x) \in C$. Thus, $c = f(x) \in C$.) Therefore, we only need to show that $C \subseteq f(f^{-1}(C))$. Let $c \in C$. The fact that f is surjective implies that there exists an $x \in X$ so that $f(x) = c$. In particular, this shows that $x \in f^{-1}(\{c\})$. Therefore, applying f to x gives c. Thus, $c \in f(f^{-1}(\{c\})) \subset f(f^{-1}(C))$. Hence we have that $C \subseteq f(f^{-1}(C))$. ■

5. Suppose $f : X \to Y$, $g : Y \to Z$, and $h = g \circ f$.

(a) Prove that if h is injective, then f is injective.

Proof: Let x_1 and x_2 be in X so that $f(x_1) = f(x_2)$. Applying the function g to each side of the equality we have $g(f(x_1)) = g(f(x_2))$, i.e., $h(x_1) = h(x_2)$. Since h is assumed to be injective, we have that $x_1 = x_2$ and hence f is injective. ■

(b) Prove that if h is surjective, then g is surjective.

Proof: Let $z \in Z$. The fact that h is surjective gives that there exists an $x \in X$ so that $h(x) = z$. Rewriting this using the definition of h we have $g(f(x)) = z$. In particular, we see that $f(x) \in Y$ and maps to z under g, so g is surjective. ■

(d) Prove or give a counterexample: if h is bijective, then f and g are bijective.

This statement is false. Let $X = \{1\}$, $Y = \{1, 2\}$, and $Z = \{1\}$. Define $f(1) = 1$, $g(1) = 1$, and $g(2) = 1$. The function h is just the function sending 1 to 1, so it is bijective. (If you don't see this, think about the definitions...) However, f is not surjective so it cannot be bijective and likewise g is not injective so cannot be bijective.