Sample homework solutions for 2.5

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1. Give an algebraic verification that if $|x-y| = |x+y|$, $x, y \in \mathbb{R}$, then either $x = 0$ or $y = 0$.

Proof: Let $x, y \in \mathbb{R}$ so that $|x-y| = |x+y|$. Squaring both sides removes the absolute values (remember these are real numbers, so the square of a real number is positive!) and so we have $(x-y)^2 = (x+y)^2$. Equivalently, $-2xy = 2xy$, i.e., $4xy = 0$. Thus, $x = 0$ or $y = 0$. ■

3. Prove Lemma 5.3.

Proof: We proved this in class, please make sure you understand the proof! ■

5. Let $f(z) = \zeta z + c$, $\zeta = e^{i\theta} \neq 1$.

(a) Show that $p = c/(1-\zeta)$ is a fixed point of f.

Proof: This is merely plugging in p to $f(z)$ and simplifying to see you get p back. Alternatively, one can solve the equation $f(p) = p$ and see that one gets $p = c/(1-\zeta)$. This second approach would allow you to find the fixed points even if they were not given, while the first method just allows you to verify what a fixed point is if someone gives you one. ■

(b) Show that $f(z) - p = \zeta(z-p)$, and conclude that f is a rotation of $\mathbb{C}$ about p through an angle of θ.

Proof: The first part of this problem is again a direct verification.

$$
\begin{aligned}
f(z) - p &= \zeta z + c - p \\
&= \zeta z + c - \frac{c}{(1-\zeta)} \\
&= \zeta z + \frac{c(1-\zeta)}{(1-\zeta)} - \frac{c}{(1-\zeta)} \\
&= \zeta z + \frac{(c - c\zeta - c)}{(1-\zeta)} \\
&= \zeta z - \zeta \frac{c}{(1-\zeta)} = \zeta(z-p).
\end{aligned}
$$

For the last part, if you do not see that this is a rotation around p, you should pick some specific examples and work them out. In particular, we know that $f(z) = \zeta z$ is a rotation around the point 0. By putting the p into the equation you are merely translating the 0 over to the point p and then rotating around the point p. ■

7. Find formulas for the following isometries $f : \mathbb{C} \to \mathbb{C}$.

(a) rotation about $2+3i$ through angle $\pi/3$.

The formula for this one follows directly from problem 5(b). Just substitute $p = 2+3i$ and $\zeta = e^{\frac{\pi i}{3}}$. So the formula is

$$f(z) = e^{\frac{\pi i}{3}}(z - (2+3i)) + (2+3i).$$

(b) reflection in the line $5z + (3-4i)\overline{z} = -4+2i$.

We use Proposition 5.6 for this problem. In particular, we know our isometry is given by $f(z) = \zeta\overline{z} + c$. Let $z = x+iy$ be on the given line. Then we have

$$\begin{aligned} -4+2i &= 5(x+iy) + (3-4i)(x-iy) \\ &= 5x + 5yi + 3x - 4y - 3yi - 4xi \\ &= (8x-4y) + (2y-4x)i \end{aligned}$$

Equating real and imaginary parts we get that $y = 2x+1$. So now that we know what line we are working with, we can easily solve the problem. We know that $z = i$ and $z = -1/2$ are both points on this line. Therefore, they are fixed points of our isometry (it reflects over the line, so it must leave the line fixed.) So we have

$$\begin{aligned} -\frac{1}{2} &= f(-\frac{1}{2}) = -\frac{1}{2}\zeta + c \\ i &= f(i) = \overline{i}\zeta + c = -i\zeta + c. \end{aligned}$$

Subtracting these equations we get that

$$-\frac{1}{2} - i = -\frac{1}{2}\zeta + i\zeta = \zeta(-\frac{1}{2} + i),$$

i.e.,

$$\zeta = \frac{-\frac{1}{2} - i}{-\frac{1}{2} + i} = -\frac{3}{5} + \frac{4}{5}i.$$

Now plug this back into the equation $i = -i\zeta + c$ and solve for c to obtain

$$c = -\frac{4}{5} + \frac{2}{5}i.$$

Thus, our isometry is given by

$$f(z) = \left(-\frac{3}{5} + \frac{4}{5}i\right)\overline{z} + \left(-\frac{4}{5} + \frac{2}{5}i\right).$$

(c) glide reflection with axis $z + \overline{z} = 2$ and translation vector $3i$.

Note that for a glide reflection one always has the axis of reflection parallel to the translation vector, so in this case we see the axis is going to have to be a vertical line. Set $z = x + iy$. Then we have

$$\begin{aligned} 2 &= z + \overline{z} \\ &= (x + iy) + (x - iy) \\ &= 2x \end{aligned}$$

i.e., the axis of reflection is the line $x = 1$. This makes the problem much easier as we know the isometry must be of the form $f(z) = \zeta\overline{z} + c$ and we can determine where points map now. In particular, we know that 1 must stay on the line $x = 1$ and translate up by $3i$, so it must map to $1 + 3i$. Similarly, we know that $1 + i$ must map to $1 + 4i$. Therefore we have the equations

$$\begin{aligned} 1 + 3i &= f(1) = \zeta \cdot 1 + c \\ 1 + 4i &= f(1 + i) = \zeta(1 - i) + c. \end{aligned}$$

Solving these simultaneous equations we obtain $\zeta = -1$ and $c = 2 + 3i$. Thus our map is $f(z) = -\overline{z} + (2 + 3i)$.