Sample homework solutions for 2.4

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1. Solve the following equations using Theorem 4.1:
(d) $(x^2 + 3x + 4)(x^3 + 3x + 5) = 6$

Note there that when one is solving such an equation, one needs to have 0 on one side to use the fact that $\mathbb{C}$ is an integral domain. It is a common mistake to think that one can leave the 6 on one side.

If we multiply this out we end up with a quartic equation, which we do not know how to solve. Therefore we must find another method. Let $z = x^2 + 3x + 4$. Then the equation reads

$$z(z + 1) = 6,$$

i.e., $z^2 + z - 6 = 0$. This factors as $(z + 3)(z - 2) = 0$. Thus we have that $z = -3$ or $z = 2$. Now we have a pair of quadratic equations:

$$x^2 + 3x + 4 = -3$$

and

$$x^2 + 3x + 4 = 2$$

Working with the first we see that we need to solve the equation $x^2+3x+7 = 0$. Applying the quadratic equation we obtain

$$x = \frac{-3 \pm i\sqrt{19}}{2}.$$

Similarly, the second equation leads to the equation $x^2+3x-2 = 0$. Applying the quadratic equation we obtain

$$x = \frac{-3 \pm \sqrt{17}}{2}.$$

Therefore, we have obtained all 4 solutions to the original equation.

2. (c) A jogger ran 10 miles in a certain amount of time. to cover the same distance in 1/4 hour less, she must run 2 m.p.h. faster. How fast did she run, and for how long?

Let r be the original speed in m.p.h. and t be the original amount of time in hours. We have the following two equations:

$$\begin{aligned} 10 &= rt \\ 10 &= (r+2)(t-\frac{1}{4}). \end{aligned}$$

Solving the first equation for r and plugging into the second equation we are led to the quadratic equation

$$4t^2 - t - 5 = 0.$$

This has solutions

$$t = \frac{1 \pm \sqrt{21}}{8}.$$

Since we need a positive time, we choose the solution $t = \dfrac{1+\sqrt{21}}{8}$ hours. The rate is then given by dividing 10 by t.

6. Solve the following cubic equations:

(b) $z^3 - 9z^2 + 9z - 8 = 0$

We begin by noting that if we make the substitution $z = x + 3$ that our polynomial reduces to

$$x^3 - 18x - 35.$$

Now we use Theorem 4.3 to solve this cubic. In this case $p = -18$ and $q = -35$, so $A = 27$. Therefore, the cube roots of A are $3, 3e^{\frac{2\pi i}{3}}, 3e^{\frac{4\pi i}{3}}$, i.e., $3, -\frac{3}{2} + i\frac{3\sqrt{3}}{2}, -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$. Therefore, the three roots of $x^3 - 18x - 35 = 0$ are $5, -\dfrac{5}{2} + \dfrac{\sqrt{3}}{2}i, -\dfrac{5}{2} - \dfrac{\sqrt{3}}{2}i$. Using the fact that $z = x + 3$, we obtain the solutions to the original equation:

$$\begin{aligned} z_1 &= 8 \\ z_2 &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ z_3 &= \frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$

9. Make addition and multiplication tables for

$$(\mathbb{Z}/2\mathbb{Z})\,[\alpha] = \{\overline{0}, \overline{1}, \alpha, \alpha + \overline{1}\}$$

where by definition arithmetic is done in $\mathbb{Z}/2\mathbb{Z}$ according to each of the following rules:

(a) $\alpha^2 = \alpha + \overline{1}$

Decide in each case whether or not $(\mathbb{Z}/2\mathbb{Z})\,[\alpha]$ is a field.

$+$	$\overline{0}$	$\overline{1}$	α	$\alpha+\overline{1}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	α	$\alpha+\overline{1}$
$\overline{1}$	$\overline{1}$	$\overline{0}$	$\alpha+\overline{1}$	α
α	α	$\alpha+\overline{1}$	$\overline{0}$	$\overline{1}$
$\alpha+\overline{1}$	$\alpha+\overline{1}$	α	$\overline{1}$	$\overline{0}$

$\cdot$	$\overline{0}$	$\overline{1}$	α	$\alpha+\overline{1}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	α	$\alpha+\overline{1}$
α	$\overline{0}$	α	$\alpha+\overline{1}$	$\overline{1}$
$\alpha+\overline{1}$	$\overline{0}$	$\alpha+\overline{1}$	$\overline{1}$	α

This is in fact a field. One can see this by observing that it is clearly a commutative ring and each element has an inverse. It may be useful in the future to note that this field is the "same" as the field $(\mathbb{Z}/2\mathbb{Z})\,[x]/(x^2-x-1)$. This will be useful possibly in chapter 3. In general, given a ring R and a polynomial $f(x) \in R[x]$, one has that $R[x]/(f(x))$ is a field if and only if $f(x)$ is an irreducible polynomial over R. This may make no sense right now, but it will be useful to look back over once we have discussed polynomial rings.