MATH 580 — SECOND MIDTERM EXAM

February 27, 2006

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 10 problems.
3. Do not separate the pages of the exam.
4. Your proofs should be neat and legible. You may and should use the back of pages for scrap work.
5. If you are unsure whether you can quote a result from class or the book, please ask.
6. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	12	
2	12	
3	11	
4	12	
5	6	
6	6	
7	8	
8	12	
9	10	
10	8	
TOTAL	100	

1. (3 points each) **(a)** Define the term "integral domain":

(b) Define the term "field":

(c) Give an example of an integral domain that is not a field. (Justify your answer!)

(d) Give an example of a ring that is not an integral domain. (Justify your answer!)

2. (3+4+4+4 points) Let $z, w \in \mathbb{C}$.

(a) Define $\overline{z}$.

(b) Prove that $\overline{z+w} = \overline{z} + \overline{w}$.

(c) Prove that $|z|^2 = z\overline{z}$.

(d) Prove that $z = \overline{z}$ if and only if $z \in \mathbb{R}$.

3. (3+3+3+2 points) Let R be a ring with the following addition and multiplication tables:

+	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	d	e	f	a
c	c	d	e	f	a	b
d	d	e	f	a	b	c
e	e	f	a	b	c	d
f	f	a	b	c	d	e

·	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	c	d	e	f
c	a	c	e	a	c	e
d	a	d	a	d	a	d
e	a	e	c	a	e	c
f	a	f	e	d	c	b

(a) Is this ring commutative? Give reasons for your answer.

(b) What are the zero-divisors in R?

(c) What are the units in R?

(d) What familiar ring is this? (No proof is required here.)

4. (6 points each)**(a)** Show that $\mathbb{Q}\left(\sqrt{7}\right)$ is a field. You may use the fact that $\mathbb{Q}\left(\sqrt{7}\right) \subset \mathbb{R}$ and $\mathbb{R}$ is a ring.

(b) Prove that $\mathbb{Q} \subset \mathbb{Q}\left(\sqrt{7}\right)$ but $\mathbb{Q} \neq \mathbb{Q}\left(\sqrt{7}\right)$.

5. (6 points) Prove that between any two distinct real numbers there are infinitely many rational numbers. You may use the fact that between any two distinct real numbers there is at least one rational number.

6. (6 points) Show that additive inverses in a ring are unique.

7. (3+5 points) **(a)** List the n^{th} roots of unity.

(b) Show that the sum of the n^{th} roots of unity is 0.

8. (4 points each) **(a)** Show that $|e^{i\theta}| = 1$ for any $\theta \in \mathbb{R}$.

(b) Show that the map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^{i\theta} z$ for some fixed $\theta \in \mathbb{R}$ is an isometry.

(c) Let $g : \mathbb{C} \to \mathbb{C}$ be an isometry with $g(i) = 1$. Show that g maps the circle of radius 6 centered at i to the circle of radius 6 centered at 1.

9. (5 points each)**(a)** Find the cube roots of 27.

(b) Using part (a) and the formula from class, show that the roots of $z^3 - 9z - 28 = 0$ are $4, -2 + \sqrt{3}i$, and $-2 - \sqrt{3}i$.

10. (8 points) Let R be a ring and $a \in R$ a fixed element. Let $S = \{ra : r \in R\}$, i.e., S is the set of multiples of a. Show that S is a subring of R not necessarily with multiplicative identity. Determine when $1_R \in S$.