MATH 580 — FINAL EXAM

March 15, 2006

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 8 problems.
3. Do not separate the pages of the exam.
4. Your proofs should be neat and legible. You may and should use the back of pages for scrap work.
5. If you are unsure whether you can quote a result from class or the book, please ask.
6. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	18	
2	14	
3	5	
4	5	
5	18	
6	10	
7	15	
8	15	
TOTAL	100	

1. (3 points each) **(a)** Let F be a field and $f(x) \in F[x]$. Define the splitting field of $f(x)$.

(b) Let X, Y, and Z be sets with $Z \subset Y$ and $g : X \to Y$ a map. Define $g^{-1}(Z)$.

(c) Define what it means for $f(x) \in F[x]$ to be irreducible.

(d) Define the term field.

(e) Let a and b be integers. Define the greatest common divisor of a and b.

(f) Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. Define $a \equiv b (\operatorname{mod} n)$.

2. (2 points each) Give examples of the following. These are no partial credit, so no explanation is necessary.

(a) An integral domain that is not a field.

(b) A field with finitely many elements.

(c) A field with infinitely many elements.

(d) An infinite ring that is NOT an integral domain.

(e) An integral domain that is NOT a field but contains $\mathbb{Q}$. (Think Chapter 3!)

(f) An injective function that is NOT surjective.

(g) A surjective function that is NOT injective.

3. (5 points) Let $z \in \mathbb{C}$. Recall that we denoted the real part of z by $\mathrm{Re}(z)$. Prove that

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}.$$

4. (5 points) Let R be an integral domain with $\mathbb{Z} \subset R$. Show that if $2^5 x = 2^3 y$, then $4x = y$.

5. (6 points each) There are 6 separate questions in this problem. Pick any three of them that you choose and ignore the other three. Please indicate CLEARLY which three you want graded, otherwise I will grade the first three.

(a) Prove or disprove: If $a|(b+c)$ then $a|b$ or $a|c$.

(b) Let p be a prime number. Prove that if $p|(a_1 a_2 \cdots a_n)$, then $p|a_i$ for some $1 \leq i \leq n$. (You may use the fact that if $p|ab$ then $p|a$ or $p|b$.)

(c) Let $f(x) \in \mathbb{Q}[x]$. Prove that if you divide $f(x)$ by $(x-2)$ then you obtain a remainder of $f(2)$. (This requires a proof, it is NOT acceptable to simply say "This is true by Proposition....")

(d) Prove that if $f(x) \equiv g(x) (\bmod p(x))$ and $g(x) \equiv h(x) (\bmod p(x))$, then $f(x) \equiv h(x) (\bmod p(x))$.

(e) Prove that $\mathbb{Q}[\sqrt{-3}]$ is a field. You may use the fact that $\mathbb{Q}[\sqrt{-3}] \subset \mathbb{C}$ and $\mathbb{C}$ is a field.

(f) Suppose $f : X \to Y$, $g : Y \to Z$, and $h = g \circ f$. Prove that if h is surjective, then g is surjective.

6. (3+7 points) **(a)** State the fundamental theorem of algebra.

(b) Use induction and the fundamental theorem of algebra to prove that if $f(x) \in \mathbb{C}[x]$, then $f(x)$ can be factored into linear factors.

7. (3 points each) Let $R = (\mathbb{Z}/11\mathbb{Z})\,[x]/(x^3 + \overline{3})$.

(a) Is R a field? Justify your answer!.

(b) Compute $\overline{5x^2 + 7x + 4} + \overline{10x^2 - 3x + 1}$.

(c) Compute $(\overline{2x^2 + 3}) \cdot (\overline{x^3 + 5x^2 + 6})$.

(d) Find a polynomial $r(x)$ of degree less then or equal to 2 so that $\overline{g(x)} = \overline{r(x)}$ where $g(x) = x^6 + 10x^3 + 5$.

(e) How many elements are in the ring R?

8. (3+4+8 points) **(a)** Write down the 6^{th} roots of unity.

(b) List the roots of $f(x) = x^6 - 2$.

(c) Find the splitting field of $f(x)$. Be sure to prove the field you find is the splitting field. In particular, your splitting field should NOT be in terms of elements that are written with e's or π's in them..