Partitions of Numbers and Sets

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The partition of a number was an early concept of number theorists. Euler was the first to offer significant insight and the concept has found its way into modular functions, set algebra, and computer science theory. The complex study of this simple idea has given birth to new analytical method, as well as more precise calculation of old method. Some of the greatest mathematical minds of the past three centuries have studied partitions, and these mathematical subdivisions have been made home to inspiring mathematical thought.

1. Introduction and Euler's Function

A partition of a number n is a way of writing n as a sum of positive integers, order being irrelevant. As an example, the partitions of 6 are:

6	3+2+1
5+1	3+3
4+1+1	2+1+1+1
4+2	2+2+1+1
3+1+1+1	2+2+2
3+2+1	1+1+1+1+1+1
3+3	

Thus, in proper notation, p(6) = 11, where p(n) is the number of partitions of n. These sets constitute solutions of the Diophantine Equation

$$1j_1 + 2j_2 + 3j_3 + \ldots + nj_n = n$$

As an example, the partitions of 3, given by (3), (2+1), (1+1+1) correspond to the solutions $(j_1, j_2, j_3) = (3,0,0), (1,1,0)$, and $(0,0,1)$, respectively, that is, (3) is *one 3*, (2+1) is *one 2 and one 1*, and (1+1+1) is *three 1's*.

In the late 18th century, Leonhard Euler derived his noted function

$$\phi(q) = \prod_{k=1}^{\infty} (1-q^k).$$

Taking a Maclaurin series of the reciprocal gives a generating function for partitions:

$$\frac{1}{\phi(q)} = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)$$

$$= (1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^3 + x^6 + x^9 + \ldots) \ldots .$$

From here, we see that the coefficient of the x^n term of the product is a count of the number of partitions of n. Thus

$$\frac{1}{\phi(q)} = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right) = \sum_{x=0}^{\infty} p(n)x^n$$

2. Representations

In the late 19th century, Norman Macleod Ferrers, out of Cambridge University, introduce his Ferrers Graphs as a way to represent partitions. These graphs are simply graphic representations of partitions of n using dots so as to visualize the partition. A Ferrers Graph for 9 is

•   •   •   •

•   •   •

•

•

which represent the partition p(9) = 4 + 3 + 1 + 1. A *conjugate* Ferrers graph would be obtained by inverting rows and columns, thus the conjugate for p(9) = 4 + 3 + 1 + 1 would be p(9) = 4 + 2 + 2 + 1. These visuals helped determine some partition equalities such as:

The number of self-conjugate partitions is the same as the number of partitions with distinct odd parts.

The number of partitions of n into no more than k parts is the same as the number of partitions of n into parts no larger than k.

The number of partitions of n into no more than k parts is the same as the number of partitions of n+k into exactly k parts.

The number of self-conjugate partitions is the same as the number of partitions with distinct odd parts.

and $\binom{m+n}{m} = \binom{m+n}{n}$.

In 1900, Alfred Young, also out of Cambridge University, took these graphs a step farther introducing the Young Tableau. In these, Young replaced dots with boxes, and in each box he placed the number equal to "all boxes that are in the same row to the right of it plus those boxes in the same column below it, plus one (for the box itself)." This number is called the hook length of box x, represented as hook(x). So for the previous example, the Young Tableau is

7	4	3	1
6	2	1	
2			
1			

The number of different Young Tableaux that can be created from a number is equal to the dimension of the irreducible representation π_λ corresponding to a partition λ. This yields the equation

$$\dim \pi_\lambda = \frac{n!}{\prod_{x\in\lambda} hook(x)}$$

In the example, the dimension would be

$$\dim \pi_\lambda = \frac{9!}{1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 7} = 180.$$

From this data, one can obtain Young Symmetrizers, which hold applications in group algebra of symmetric groups.

3. A Generating Function

Moving beyond foundations comes the Indian genius Srinivasa Ramanujan. Contriving monumental results from the most basic of beginnings, he was taken out of impoverished India to study advanced mathematics with renowned number theorists in England. Among his many startling results was an asymptotic expression for the partition of an integer. In 1918, using elliptic modular functions, Ramanujan, along with his mentor in England G. H. Hardy, obtained an approximation for the partition function:

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}} \text{ for large n.}$$

While not exact, the error was incredibly small, and for the first time, insight was gained into large partition numbers. By hand, partition numbers became unwieldy much too quickly, evident in $p(1000) \approx 2.4 \times 10^{31}$. From these studies, Ramanujan found some elementary congruences:

$$p(5k+4) \equiv 0 \pmod 5$$
$$p(7k+5) \equiv 0 \pmod 7$$
$$p(11k+6) \equiv 0 \pmod{11}$$

Checking the first few cases of each, we see

$$p(9) = 30 \equiv p(14) = 135 \equiv p(19) = 490 \equiv 0(\bmod 5)$$
$$p(12) = 77 \equiv p(19) = 490 \equiv p(26) = 2436 \equiv 0(\bmod 7)$$
$$p(17) = 297 \equiv p(28) = 3718 \equiv p(39) = 26015 \equiv 0(\bmod 11)$$

With the advent of computing it was later shown that there are congruences of this form modulo p, where p is any prime, and also modulo x, where x is any number coprime to 6.

In discovering this approximation, Ramanujan and Hardy, along with John Littlewood would give rise to the circle method. In it, one studies an integral of the form

$$I_n = \int f(z)x^{-(n+1)}dz = 2\pi i a_n$$, where a_n is a sequence.

This contour integral is over a unit circle centered at 0 and is convergent at r = 1. However, since f is not defined at r = 1, analysis proves problematic. This is where the circle method is implemented. By using the Farey Sequence, defined as the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to *n*, arranged in order of increasing size, one can analyze singularities on the edge of the unit circle (r = 1). These 'roots of unity', expressed by the equation

$$\zeta = e^{\left(\frac{2\pi i r}{s}\right)}$$, with r/s in lowest terms,

when applied on the contour integral determine the singular behavior. Implementation of the circle method has been used in other areas of interest, including further study of Diophantine equations via the Hasse Principle. The Hasse Principle states

An equation can be solved over the rational numbers if and only if it can be solved over the real numbers and over the p-adic numbers for every prime p.

p-adic being an arithmetic extension tool. The circle method was used in the proof of this statement, which has been shown quite important in rational solvability.

Working with Ramanujan's partition studies, in 1937 a German mathematician, Hans Rademacher, found an exact formula for partitions of n. From Hardy and Ramanujan's initial work, a more exact approximation for p(n) was given as

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k \le \alpha\sqrt{n}} A_k(n)\sqrt{k}\,\frac{\partial}{\partial n}\left(\frac{e^{\frac{C\sqrt{n-\frac{1}{24}}}{k}}}{\sqrt{n-\frac{1}{24}}}\right) + O(n^{-\frac{1}{24}})$$

with α arbitrary, $A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k} e^{-\frac{2\pi i h n}{k}}, c = \pi\sqrt{\frac{2}{3}}$,

and $\omega_{h,k}$ are "24kth roots of unity which have their origin in the theory of the transformation of modular fractions."

In this equation, O represents the error of the function, which tends to zero as n approaches infinity. Rademacher improved this formula, observing that hyperbolic sines can give an exact solution. Thus the equality for partition numbers is given as

$$p(n)=\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}A_k(n)\sqrt{k}\frac{\partial}{\partial n}\left(\frac{\sinh\frac{C\sqrt{n-1/24}}{k}}{\sqrt{n-1/24}}\right).$$

This, of course, is by no means a "simple" equation. In fact, even with modern computing, a 4 digit, prime n-value is beyond conventional algorithms. The RSA, during the 90's, threw down the gauntlet challenging theorists to factor 100 - 500 digit partition numbers, offering handsome cash rewards. In doing so, advancements in cryptography have been made in terms of prime factorization method.

4. Partitions of Sets

This concept of partitioning can also be applied to sets. To define, a partition of a set *X* is a set of nonempty subsets of *X* such that every element *x* in *X* is in exactly one of these subsets. In more detail, a set A of subsets of B is a partition of B if

1. No element of A is empty
2. The union of the elements of A is equal to B.
3. The intersection of any two elements in A is empty.

For instance, the set A = {x, y, z} has the five partitions

{{x},{y},{z}}
{{x, y},{z}}
{{x},{y,z}}
{{x, z},{y}}
{{x, y, z}}

These partitions are used in set algebra, for instance in forming a basis of symmetric polynomials from elementary polynomials. Symmetric polynomials hold applications in linear algebra and polynomial rooting, among other disciplines. One can form this basis as follows:

Define elementary polynomials, $e_n(x_1,x_2,...,x_n)=x_1x_2\cdots x_n$, as

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots$$
$$e_r = m_\lambda = unitpolynomial$$

Then

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu}m_\mu$$, with μ a positive integer,

gives rise to the symmetric basis

$$\{e_\lambda, \lambda, m\}$$

Also, in dealing with the partitioning of sets arises 'the partition problem' in computer science. Related to the infamous 'P versus NP' Millennium problem, the partition problem is "to decide whether a given multiset of integers can be partitioned into two "halves" that have the same sum. More precisely, given a multiset S of integers, is there a way to partition S into two subsets S_1 and S_2 such that the sums of the numbers in each subset are equal?" Since this problem is basically equivalent to asking "does any subset of integers less than n sum to zero modulo n?" partitioning of integers becomes crucial. Computer science theorists use this to determine whether solutions that can be verified in polynomial time also can be computed in polynomial time (P versus NP).

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