INTEGER PARTITIONS

ABSTRACT. This paper consists of a basic overview of partition theory. It begins with the definition of a partition, a discussion of the partition function, $p(n)$, the intermediate function, $p(k, n)$, and generating functions for $p(n)$. Also covered are Ferrers graphs, Ramanujan congruences, rank and crank.

1. INTRODUCTION

Where primality deals with breaking whole numbers into products, partition theory deals with how integers can be broken down into a sums. Much of what is known about partition theory began Euler and grew with the collaboration of Srinivasa Ramanujan, the Indian mathematical prodigy, and G. H. Hardy at Cambridge [1].

Definition 1.1. A *partition* of a positive integer n is defined to be a sequence of positive integers whose sum is n [1]. The order of the summands is unimportant when writing the partitions of n, but for consistency, partitions of n will be written with the summands in a non-increasing order. For example, the partitions of $n = 4$ are given as:

$$4 = 4$$
$$= 3 + 1$$
$$= 2 + 2$$
$$= 2 + 1 + 1$$
$$= 1 + 1 + 1 + 1$$

Another way to define a partition of a positive number n is as a solution of the diophantine equation [2]:

$$1 * x_1 + 2 * x_2 + 3 * x_3 + ... + n * x_n = n$$

where the partitions of n include the sums corresponding to the solutions.

Definition 1.2. A summand in a partition is called *part*. In the partition 3+1, we have that 3 and 1 are parts comprising the partition.

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Definition 1.3. The *partition function* $p(n)$ counts the number of unique partitions of the positive integer n. Recall that there were 5 unique partitions of 4. Thus, $p(4) = 5$. The value of $p(n)$ is shown below for $0 \leq n \leq 10$:

$$p(0) = 1$$
$$p(1) = 1$$
$$p(2) = 2$$
$$p(3) = 3$$
$$p(4) = 5$$
$$p(5) = 7$$
$$p(6) = 11$$
$$p(7) = 15$$
$$p(8) = 22$$
$$p(9) = 30$$
$$p(10) = 42$$

Definition 1.4. The *intermediate function* is given as $p(k,n)$. The intermediate function is defined such that it counts the partitions of n with the largest added being no smaller than k [4]. Some values of $p(k,n)$ are given below [4]:

$$p(1,4) = 5$$
$$p(2,8) = 7$$
$$p(3,12) = 9$$
$$p(4,16) = 11$$
$$p(5,20) = 13$$
$$p(6,24) = 16$$

2. How Many Ways Can n Be Partitioned?

We know that $p(n)$ counts the number of ways n can be partitioned, but how can we calculate $p(n)$? There is a clear brute force algorithm that can determine the number of partitions for n simply by listing the number of partitions and then counting them as we did in the definition of a partition, but $p(n)$ grows quite quickly. While $p(n)$ starts small, the number of partitions of 100, a relatively small number, is 190,569,292.

It's easy to see that listing partitions and counting is not going to be an effective method for determining $p(n)$. A numerical formula or generating function, is much more useful for calculating large values of $p(n)$.

One such formula for $p(n)$ was given by Hardy and Ramanujan and then independently by J. V. Uspensky:

$$p(n) \approx \frac{e^{c*\sqrt{n}}}{4*n*\sqrt{3}}$$

where c is given as:

$$c = \pi * (2/3)^{1/2}.$$

This is, however, just an approximation to $p(n)$. While very near the actual value of $p(n)$, it still does not provide us with the exact value we desire [1].

Hans Rademacher provided another way to determine $p(n)$, though [5]. It is given by a series called the Rademacher's series that comes from the reciprocal of Euler's function:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} \right)$$

which can be expanded to

$$(1 + x + x_2 + x_3 + ...)(1 + x_2 + x_4 + x_6 + ...)(1 + x_3 + x_6 + x_9 + ...)...$$

The x_n term gives us the number of partitions of n [4] [5]. This, however, is not a formula for $p(n)$, but rather a generating function which allows $p(n)$ to be calculated exactly for any n [4].

It is also interesting to note that the terms of Rademacher's series can be truncated to provide approximations of $p(n)$. By taking just the first term in the series for $n = 200$, the result approximates $p(n)$ with an error of 0.004 [1]. This is quite helpful when only the order of magnitude of $p(n)$ is required.

3. A Closer Look at the Intermediate Function

The intermediate partition function is also of interest in partition theory. In order to determine the value of $p(k, n)$, we first note that we can break the partitions counted by $p(k, n)$ into two distinct, mutually exclusive groups:

1. Partitions where the smallest summand is k
2. Partitions where the smallest summand is greater than k

Looking first at the partitions where the smallest summand is k, we have that $p(k, n - k)$ gives the correct number of partitions. This follows because we can form a partition of $n - k$ with smallest addend k and then add k to each part in the partition to obtain a sequence summing to n.

Then, looking at the second condition, we obtain that the number of partitions meeting the condition number $p(k + 1, n)$ since this means the least addend will be at least $k + 1$ which is strictly greater than k.

Because we had that the two groups of partitions were mutually exclusive, the number of partitions satisfying either condition and thus equal to $p(k, n)$ is $p(k + 1, n) + p(k, n - k)$. After handling the base cases of $p(k, n) = 0$ for $k > n$ and $p(k, n) = 1$ for $k = n$, we get a recursively defined value for $p(k, n)$ [4].

4. FERRERS DIAGRAMS

Definition 4.1. A Ferrers diagram (or Ferrers graph), after Norman Macleod Ferrers, is a tool that can be used to make working with partitions easier [4]. As described by Herbert S. Wilf:

> T"he Ferrers diagram of an integer partition gives us a very useful tool for visualizing partitions, and sometimes for proving identities. It is constructed by stacking left-justified rows of cells, where the number of cells in each row corresponds to the size of a part. The first row corresponds to the largest part, the second row corresponds to the second largest part, and so on" [6].

An example of a Ferrers diagram for a partition of 9 is given below:
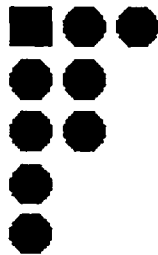
The graph above corresponds to the partition 5+3+1. Note that Ferrers diagrams are drawn using dots. If blocks are used, the diagrams are known as Young diagrams or Young graphs.

Definition 4.2. If we take all columns and make them rows while making all rows columns, we obtain another partition of 9:

■ ● ● ● ●
● ● ●
●
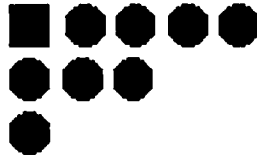
This operation is known as the *conjugate diagram* of the Ferrers graph for 5+3+1 [7]. This is sometimes called the *dual partition* [8]. If the graph is visualized as a matrix, taking the transpose of the matrix is analogous to taking the conjugate of the graph.

Definition 4.3. A graph whose conjugate graph is identical to the graph itself is said to be *self-conjugate* [9]. An example of a self-conjugate graph is shown below:

■ ● ●
●
●
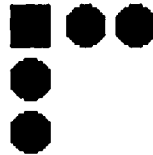
Ferrers Diagrams will allow us to prove some trivial facts about partitions which would otherwise be difficult to prove.

Theorem 4.4. *The number of partitions of k into j parts is equal to the number of partitions of k such that the largest part has size j* [6] [7].

Proof:
Because we can take any partition of k with j parts and draw a Ferrers diagram and then find the conjugate diagram (and vice-versa), we can establish a bijection between the two sets. Thus, they must have the same number of elements [6] [7].

Theorem 4.5. *The number of partitions of k such that no two parts have the same size is equal to the number of partitions of k such that no size is skipped from 1 to some maximum size j* [7].

Proof:
"A partition with no two parts of the same size has a Ferrers diagram where each row is of a different size. A partition with no size skipped (starting from size one) has a Ferrers diagram where each column is of a different size" [7]. We can form a bijection between the tow types of graphs using conjugate graphs [7].
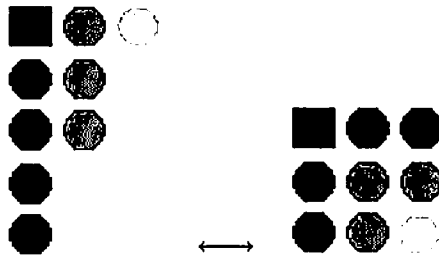
Theorem 4.6. *The number of partitions containing only odd parts is equal to the number of partitions with distinct parts for all positive integers* [9].

Proof:
Note that each column with an odd number of elements can be "folded" to obtain a self-conjugate graph as shown below [4]:
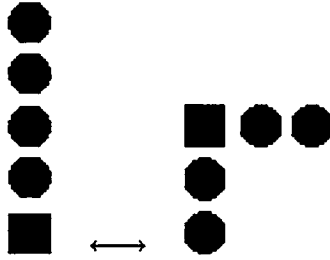
A bijection can then be obtained between the two types of partitions as shown below [4]:

Thus, the two types of partitions have the same number of elements.

5. RAMANUJAN CONGRUENCES

In 1919, Ramanujan discovered several properties of the partition function in modulo and made a famous conjecture about a generalized congruence relation [1].

Conjecture 5.1. (Ramanujan) For $q = 5, 7$, or 11, if $24*n \equiv 1 \pmod{q^k}$, then $p(n) \equiv 0 \pmod{q^k}$ for all $k \geq 0$ [1].

This conjecture was later shown to be false when $k = 3$, $n = 243$ and $q = 7$ because

$$24 * n = 24 * 243 = 5832 \equiv 1 \pmod{7^3}$$

but

$$p(243) = 1339978259344888 \equiv 245 \not\equiv 0 \pmod{7^3}$$

so the conjecture fails. But, Ramanujan proved several congruence relations in specific moduli [1]:

Theorem 5.2. *(Ramanujan) For any integer k, $p(5*k+4) \equiv 0(\mathrm{mod}\,5)$* [1] [3]

Example 5.3.

$$p(5*7+4) = p(39) = 31185 \equiv 0(\mathrm{mod}\,5)$$

Theorem 5.4. *(Ramanujan) For any integer k, $p(7*k+5) \equiv 0(\mathrm{mod}\,7)$* [1] [3]

Example 5.5.

$$p(7*5+5) = p(40) = 37338 \equiv 0(\mathrm{mod}\,7)$$

Theorem 5.6. *(Ramanujan) For any integer k, $p(11*k+6) \equiv 0(\mathrm{mod}\,11)$* [1] [3]

Example 5.7.

$$p(11*5+6) = p(66) = 2323520 \equiv 0(\mathrm{mod}\,11)$$

Before talking about the proofs of these congruences, we need to define the rank of a partition.

Definition 5.8. The *rank* of a partition is given as the largest term minus the number of terms in the partition [10]

The proofs of the partition congruences for 5 and 7 can be given using the rank of a partition. The first attempts to use the rank to prove Ramanujan's congruences were made by Freeman Dyson.

> "To group the partitions of 4, mathematicians divide the rank by 5, and the remainder is the grouping number. They use modular, or clock, arithmetic, to replace each negative number with the positive number with which it would share a position on the face of a clock having, in this case, five numbers. So, before being divided by 5, the rank 1 is replaced by 4 and the rank 3 is replaced by 2.
>
> After looking at many examples, Dyson made a conjectureproved in the 1950s by Atkin and Peter Swinnerton-Dyer of Cambridgethat in Ramanujan's congruences for 5 and 7, the rank divides the partitions into five and seven equal-size groups, respectively. In other words, the grouping created by the rank explains concretely why the partition numbers are divisible by 5 or 7" [10].

The proof for Ramanujan's congruence modulo 11 came using another measure of a partition used by graduate student Karl Mahlburg at The University of Wisconsin [10].

It's worthwhile to note that while it might seem like a safe guess that for any integer k

$$p(13 * k + 7) \equiv 0 \pmod{13}$$

would be true by following the pattern formed by Ramanujan's theorems, this fact does not hold [10].

Because the pattern breaks down, it was supposed for decades that Ramanujan's three partition congruences were the only congruences for prime moduli, but in 1968 another was discovered by A. Oliver L. Atkin. This, however, is much more complicated than Ramanujan's congruences as given in the next theorem [10].

Theorem 5.9. *(Atkin) For any integer k, $p(17303*k+237) \equiv 0 \pmod{13}$* [10]

Because congruences exist for 5,7,11, and 13, it is natural to wonder if partition congruences exist for more primes. Ken Ono, after having read some of Ramanujan's work proved that partition congruences exist for all primes greater than or equal to 5.

Theorem 5.10. *(Ono) For p a prime such that $p \geq 5$, a partition congruence exists modulo p.*

Since Ramanujan discovered his congruences, there have also been some similar congruences established for prime powers. An example is given in the following theorem:

Theorem 5.11. *(Ramanujan) For any integer k, $p(25 * k + 24) \equiv 0 \pmod{5^2}$* [1]

Example 5.12.

$$p(25 * 2 + 24) = p(74) = 7089500 \equiv 0 \pmod{5^2}$$

6. ARBITRARY MODULI

Though partition theory has been studied in some capacity since Euler, there is still relatively little known about partitions and the partition function. Some open questions in the field today concern $p(n)$ reduced by some arbitrary moduli. The following are a few open conjectures about $p(n)$ by Paul Erdos and Morris Newman [3].

Conjecture 6.1. (M. Newman) If M is a positive integer, then in every residue class $r \pmod M$ there are infinitely many integers N for which [3]

$$p(N) \equiv r \pmod M$$

Conjecture 6.2. (P. Erdos) If M is prime, then there is at least one non-negative integer N_M for which [3]

$$p(N_M) \equiv 0(\bmod M)$$

As noted, these are unproven conjectures and are among the unanswered questions in partition theory.

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