Introduction:

Number theory is one of the oldest and most active fields of mathematics. The beauty of the subject stems from the simplicity of many of the problems to state. Number theory is, in short, the study of numbers. In particular, we are usually interested in the integers,

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

and the rational numbers,

$$\mathbb{Q} = \{ \tfrac{a}{b} : a, b \in \mathbb{Z}, \ b \neq 0 \}.$$

One may wonder what else do we want to know about numbers; it seems, especially to nonmathematicians, that we know all there is to know about numbers. We will begin by listing a few deep problems in number theory that can be easily stated.

<u>Fermat's Last Theorem</u> (1637): Originally stated by Fermat, he claimed that the equation

$$x^n + y^n = z^n$$

has no solutions $(x,y,z) \in \mathbb{Z}^3$ with $xyz \neq 0$ and $n \geq 3$.

Of course, $n=2$ is the pythagorean theorem that all high school math students learn. So even in this case, we could ask for all the rational triple $(x,y,z) \in \mathbb{Q}^3$ that satisfy $x^2+y^2=z^2$.

For $n \geq 3$, this was finally proven in 1995 by Andrew Wiles, and is way beyond this course. For $n=2$, we will characterize all rational solutions later in the term.

Twin Prime conjecture:

Prime numbers p_1, and p_2 are called twin primes if $p_2 - p_1 = 2$; i.e, if p_1 and p_1+2 are both prime numbers. For example, 3 and 5, 5 and 7, 11 and 13 are all twin primes. While it is easy to show there are ∞'ly many primes, it is still not known if there are ∞'ly many twin primes!

Pell's Equation:

We would like to determine solutions $(x,y) \in \mathbb{Q}^2$

of the equation

$$x^2 - dy^2 = 1$$

for $d \in \mathbb{Z}$. It again seems like this may be an easy problem, but it turns out we must develop the notion of continued fractions to study this equation!

Congruent Numbers:

We say an integer n is a congruent # if there is a right triangle with ~~side~~ rational side lengths that has area n. This is again a very simple problem to state but requires very difficult modern mathematics to solve. We will look at this problem later in the term and see how elliptic curves arise in the solution of this problem.