Math 573 Problem Set 3

1. Prove that every integer of the form $3n+2$ has a prime factor of this form as well. Can you use this fact to prove there are infinitely many primes of the form $3n+2$?

2. (a) Prove that for any integer n, one has n^2 is equivalent to $0, 1, 4, 5, 6$, or 9 modulo 10.

(b) Find the values of n for which $1! + 2! + \cdots + n!$ is a perfect square.

3. If $a \equiv b \pmod{n}$, prove that $\gcd(a, n) = \gcd(b, n)$.

4. Determine the last three digits of 15^{799} by hand.

5. Write a short program in SAGE (or whatever computer software you use) that produces all of the solutions to the equation $ax \equiv b \pmod{n}$.

6. Recall in class that we proved that if n_1, n_2 are relatively prime positive integers, then

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \end{aligned}$$

has a unique simultaneous solution modulo $n_1 n_2$. Use this and induction to prove that if $n_1, n_2, \ldots, n_r$ are positive integers so that $\gcd(n_i, n_j) = 1$ if $i \neq j$, then the system

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_r \pmod{n_r} \end{aligned}$$

has a unique simultaneous solution modulo $n_1 n_2 \cdots n_r$.

7. Find, by hand, two incongruent solutions modulo 210 of the system

$$\begin{aligned} 2x &\equiv 3 \pmod{5} \\ 4x &\equiv 2 \pmod{6} \\ 3x &\equiv 2 \pmod{7}. \end{aligned}$$

8. Assuming that 495 divides $273x49y5$, find the digits x and y.

9. A palindrome is a number that reads the same backwards as forwards (for example, 511343115 is a palindrome.) Prove that any palindrome with an even number of digits is divisible by 11.

10. Use SAGE (or an equivalent computer program) to find all solutions to the congruence

$$x^3 + 3x^2 + 37 \equiv 0 (\mathrm{mod}\, 51).$$

11. Let $f(x)$ be a fixed polynomial with integer coefficients. For any positive integer n, let $N(n)$ be the number of solutions of the congruence $f(x) \equiv 0 (\mathrm{mod}\, n)$. If $n = n_1 n_2$ with $\gcd(n_1, n_2) = 1$, prove that $N(n) = N(n_1)N(n_2)$. Is the statement true without the condition on $\gcd(n_1, n_2)$? If so, prove it. If not, give a counterexample.

12. Recall that in class we defined the Riemann zeta function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$$

and used it to show there are infinitely many primes. We use it again here to show that the sum

$$\sum_{p} \frac{1}{p}$$

diverges where the summation is over all primes.

(a) Prove that the series $\sum_{p} \frac{1}{np^n}$ converges for $n \geq 2$.

(b) Let $\sum_{p} \frac{1}{2p^2} = A$. Show that $\sum_{p} \frac{1}{np^n} \leq \frac{A}{2^{n-2}}$. It may be helpful to look at $n = 3, 4$, and 5 to get an idea of where the power of 2 is coming from. Just look at the first term of the series and compare it to the first term of the series for $n = 2$.

(d) Conclude from parts (a) and (b) that the sum $\sum_{n=2}^{\infty} \sum_{p} \frac{1}{np^n}$ converges.

(e) Recall that the unique factorization of integers allowed us to write

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}.$$

Take logarithms of both sides of this equation to obtain

$$\begin{aligned}\log(\zeta(s)) &= -\log\left(\prod_p (1-p^{-s})\right)\\ &= -\sum_p \log(1-p^{-s})\end{aligned}$$

where we have used the properties of logarithms extended to infinite sums. Use the Taylor series for $\log(1+x)$ with $x = p^{-s}$ to obtain a double summation, one over primes p and one from 1 to ∞.

(f) Set $s = 1$ in part (d) and using part (c) and the fact that the Riemann zeta function diverges at $s = 1$ to conclude that the sum

$$\sum_p \frac{1}{p}$$

diverges.

12. A gang of 17 bandit stole a chest of gold coins. When they tried to divide the coins equally among themselves, there were three left over. This caused a fight in which one bandit was killed. When the remaining bandits tried to divide the coins again, there were 10 left over. Another fight started, and five of the bandits were killed. When the survivors divided the coins, there were four left over. Another fight ensued in which four bandits were killed. The survivors then divided the coins equally among themselves, with none left over. What is the smallest possible number of coins in the chest?