Math 573 Problem Set 2

1. Let n be an integer so that n is a perfect square and a perfect cube, i.e., there exists $a \in \mathbb{Z}$ so that $n = a^2$ and an integer $b \in \mathbb{Z}$ so that $n = b^3$. Prove that n is of the form $7k$ or $7k + 1$.

2. Let $a, b \in \mathbb{Z}$. Prove that if there exists integers m and n so that

$$\gcd(a, b) = am + bn,$$

then $\gcd(m, n) = 1$.

3. Prove that the product of any 4 consecutive integers is divisible by 24.

4. Prove there are no perfect squares in the sequence

$$11, 111, 1111, 11111, \ldots$$

(Hint: What is the remainder when you divide a term in the sequence by 4?)

5. Let n be an integer. Prove that $360 \mid n^2(n^2 - 1)(n^2 - 4)$.

6. Use the Euclidean algorithm to find the greatest common divisor of 123456 and 87654321 and write the greatest common divisor as a linear combination of 123456 and 87654321.

7. Find a prime number p with $n \leq p \leq 2n$ for $n = 975318642$.

8. If $\gcd(a, b) = 1$, then $\gcd(ac, b) = \gcd(c, b)$. (Do this without resorting to prime factorizations!)

9. A farmer purchased 100 head of livestock for a total cost of \$4000. Prices were as follow: calves, \$120 each; lambs, \$50 each; piglets, \$25 each. If the farmer obtained at least one animal of each type, how many of each did he buy?

10. Prove that if $p \geq 5$ with p prime, then $p^2 + 2$ is composite. (Hint: Consider the forms p can take when divided by 6.)

11. Compute $\pi(x)$ and $x/\log x$ for $x = 20, 20^2, 20^3, 20^4, 20^5, 20^6$. Any conjectures on $\lim_{x\to\infty} \dfrac{\pi(x)}{x/\log x}$?

12. Determine all primes of the form $n^2 - 4$.

13. Let $\{p_1, p_2, \ldots, p_n\}$ be a set of prime numbers. Prove that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}$$

is not an integer.

14. Consider the set $\mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} : a, b \in \mathbb{Z}\}$.

(a) Prove that this set is closed under addition, subtraction, and multiplication. (Closed under addition means if you take two elements in this set and add them together you get another element in this set, likewise for subtraction and multiplication.)

(b) Prove that $\mathbb{Z}$ is a subset of $\mathbb{Z}[\sqrt{-6}]$.

(c) Define the norm function $N : \mathbb{Z}[\sqrt{-6}] \to \mathbb{Z}_{\geq 0}$ by $N(a+b\sqrt{-6}) = a^2+6b^2$. Which numbers in $\mathbb{Z}[\sqrt{-6}]$ have norm equal to 1?

(d) Prove that for $\alpha, \beta \in \mathbb{Z}[\sqrt{-6}]$ we have $N(\alpha\beta) = N(\alpha)N(\beta)$.

(e) We say that an element $\alpha \in \mathbb{Z}[\sqrt{-6}]$ factors if there exist $x, y \in \mathbb{Z}[\sqrt{-6}]$ with $N(x) > 1$ and $N(y) > 1$ with $\alpha = xy$. Prove that if α factors into xy, then $1 < N(x) < N(\alpha)$.

(f) We say that an element $\alpha \in \mathbb{Z}[\sqrt{-6}]$ is prime if it cannot be factored in the sense of part (d). Prove that $2, 5, (2 + \sqrt{-6})$, and $(2 - \sqrt{-6})$ are all prime numbers in $\mathbb{Z}[\sqrt{-6}]$.

(g) Prove that $10 \in \mathbb{Z}[\sqrt{-6}]$ does not have a unique factorization into primes. Conclude that there is no corresponding Fundamental Theorem of Arithmetic for $\mathbb{Z}[\sqrt{-6}]$.