Math 573 Problem Set 1

1. Prove that 3 is a factor of $4^n - 1$ for every positive integer n.

2. Use induction to prove that for all $n \geq 1$

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

3. Prove that the cube of any integer can be written as the difference of two squares. (Hint: Observe that $n^3 = (1^3+2^3+\cdots+n^3)-(1^3+2^3+\cdots+(n-1)^3)$. Now prove that $1^3 + 2^3 + \cdots + m^3$ is a square for any integer m.)

4. Prove that $\dfrac{(2n)!}{2^n n!}$ is an integer for all $n \geq 0$.

5. Let A, B, and C be sets. Prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

6. Let A and B be subsets of C. Prove

$$C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B).$$

7. **(a)** Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

(b) Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

(c) Use part (a) and (b) to prove

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$