THE GOLDEN RAIO AND FIBONACCI

[redacted]

ABSTRACT. This paper is an investigation of the history of the golden ratio φ and its relation to various areas of mathematics and science; the insight gained from this historical undertaking is then applied to yield a solution to a centuries-old problem: determining a closed-form equation for calculating the nth tern in the Fibonacci sequence.

I. INTRODUCTION

The Golden Ratio is a transcendental number that has, for many centuries, fascinated mathematicians, scientists, and even artists. The number (also known as the Golden Section, the Golden Mean, and the Divine Proportion, among other monikers) is seemingly ubiquitous in mathematics and nature, manifesting itself in such diverse areas as geometry, fractals, Fibonacci numbers, and even phyllotaxis (the way in which plants develop leaf patterns in order to maximize sun exposure, a topic very far removed from number theory). The Golden Ratio is typically denoted φ (named for Phidias, a famous Greek sculptor who lived during the 5^{th} century B.C. and was said to embody the Golden Ratio in his work) [2] and the number is widely regarded as being aesthetically pleasing in its manifestations in both visual arts and music; as will be discussed below, the number φ is embodied in works as vastly different (chronologically and culturally) as the Aegean Acropolis and Penrose tilings.

In order to facilitate the ease with which this Paper is to be read and understood, the information presented is arranged chronologically, intermingling historical and mathematical details; such an approach is intended to better maintain the reader's interest and reinforce the idea that developments in mathematics and a thorough understanding of the properties of the Golden Ratio are intimately related concepts. Having established the requisite historical and mathematical background, the main task of applying the Golden Ratio to the problem of determining a closed form for the nth term of the Fibonacci sequence will be investigated. In the following, the historical and mathematical details pertinent to the Golden Ratio are investigated in the Classical, Medieval, and Modern periods.

II. THE BIRTH OF PHI

Although the first civilization to be aware of the Golden Ratio and to embody this proportion in its architecture and other culturally significant works is speculative and debatable, the oldest known precise definition of φ was provided in Euclid's *Elements* [2]. By "extreme and mean ratios" [3], Euclid proposed that a straight length be cut such that the ratio of the smaller piece to the larger piece is identical to the ratio of the larger piece to the entire length. In the Figure below, the ratios a/b and (a+b)/a are equal.

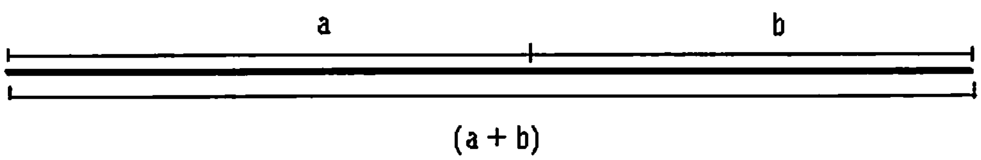

Figure 1 Dividing a line segment according to the Golden Ratio; $\varphi = a/b$

Although the majority of the mathematical properties of φ were unknown to the Classical Greeks, early mathematician-philosophers such as Pythagoras and Euclid were very interested in the role of φ in regular geometric forms, both planar and three-dimensional (i.e. what is now known as "Golden Geometry")[4]. The classical ruler-and-compass construction of φ is illustrated below. Draw a circle of unit diameter and center A, extend a line AC from the circle's center to a point C on the circle, draw a line BC of unit length perpendicular to AC, and draw a ray that begins at B, pierces the circle at D, passes through the center A, and exits the circle at point E (see Figure 2 below). Then, the ratio |BE|/|BC| (where the absolute value signs denote lengths of line segments) is the Golden Ratio, φ. Numerically, then, the value of φ can be calculated as follows: |BC| = 1 by construction; |AC| = |AD| = |AE| = ½ since each of these segments is equal to the circle's radius in magnitude.

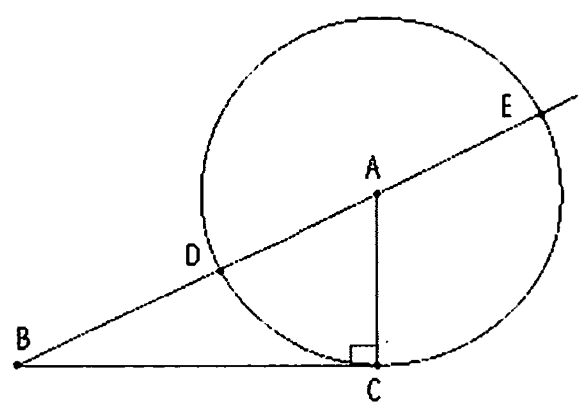

Figure 2 Using ruler and compass to derive $\varphi = |AE|/|BC$

The triangle ΔABC is a right triangle, so the Pythagorean Theorem can be invoked:

$$
\begin{aligned}
|AB| &= (|BC|^2 + |AC|^2)^{1/2} \\
&= (1 + 1/4)^{1/2} \\
&= 1/2 \cdot 5^{1/2}
\end{aligned}
$$

Then, adding, $\varphi = |BE| = |AB| + |AE| = 1/2\,(1 + 5^{1/2}) \sim 1.618033989\ldots$ Modern readers will immediately recognize that φ is an irrational number, being the sum of rational $1/2$ and irrational $(1/2 \cdot 5^{1/2})$ (since, by closure of the rational numbers, $5^{1/2}$ is irrational, so $(1 + 5^{1/2})$ and hence $1/2(1 + 5^{1/2})$ are irrational); nevertheless, a brief and straightforward proof of the irrationality of φ is provided below.

Proving that φ is an irrational number is brief but requires some algebraic manipulation; consider the illustration of Figure 1, assuming (without loss of generality) that a > b. If φ is a rational number, then

$$\phi = \frac{a+b}{a} = \frac{a}{b},$$

where the fraction a/b is assumed to be written in lowest terms. Then, multiplying through by (a·b), collecting terms, and dividing,

$$
\begin{aligned}
&b \cdot (a+b) = a \cdot a \\
&a \cdot (a-b) = b \cdot b \\
&\frac{a}{b} = \frac{b}{a-b},
\end{aligned}
$$

where (a-b) is a positive integer. Note that now the fraction a/b has been rendered as a fraction b/(a-b) with a smaller numerator and denominator; this fact contradicts the assumption that the original ration a/b was written in lowest terms. Consequently, by contradiction, $\varphi = a/b$ must be an irrational number since it cannot be written as a ratio of integers.

The simplest and most common forms explored via Golden Geometry are isosceles triangles with angles (72°, 72°, 36°); these planar geometric figures arise in regular pentagons, decagons, and pentagrams [4]. The appearance of φ in the regular pentagon is illustrated on the following page (Figure 3); connecting a vertex with the two vertices opposite creates an isosceles triangle with the desired angle measurements. The ratio of the lengths of one of the large sides to the small side and the ratio the sum of large and small sides to a large side are both equal to φ.

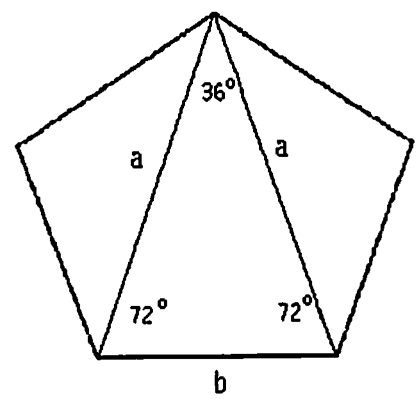

Figure 3 Constructing a Golden Triangle from a pentagon; $\varphi = a/b$

As for the cultural significance of φ in the ancient Greek world, mathematicians were not the only Greeks impressed by the simplicity and beauty of the Golden Ratio. Architects and sculptors found the ratio to be aesthetically pleasing and sought to embody the Divine Proportion in their artistic endeavors [4]. One of the most famous examples of φ's appearance in the classical Greek world is the Parthenon, one of the many structures comprising the Aegean Acropolis. According to traditional sources, the stylobate of the Parthenon (i.e. the top step of the stepped platform upon which the columns of the structure stand) incorporates several golden rectangles (rectangles with sides proportionally related by φ); however, a modern investigation of the issue raises doubt as to whether the architects of the structure truly intended for this proportionality to appear, or whether this phenomenon is the result of φ enthusiasts knowingly distorting measurements to fit their expectations [3]. On the other hand, as mentioned above, the Greek sculptor Phidias (who even predated Euclid by at least a century) is credited with having employed the Golden Ratio in the statues he constructed for the Acropolis [4].

III. PHI IN THE MIDDLE AGES

In the beginning of the thirteenth century, the mathematician Fibonacci (an abbreviated form of *Filius Bonacci*) developed a now-famous sequence bearing his surname. In this series, each term is the sum of the two previous terms (where the first two terms are taken to be 1), as follows:

$$a_n = a_{n-1} + a_{n-2}.$$

The first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … As will be discussed below (see the PHI GROWS UP section below), the Fibonacci sequence and the Golden Ratio share a deep connection; this relationship will prove quite useful when applied to the problem of finding a closed form for the nth term of the Fibonacci sequence.

Although the Middle Ages were a period best characterized as intellectual stagnation, humanity's interest in and appreciation for the Golden Ratio was revitalized during the Renaissance. It was in this period that painters and sculptors employed φ in their works as an homage to the wondrous achievements of classical Greece and due to the aesthetic appeal of the Golden Ratio. Examples from this period abound, but one work suffices to demonstrate the appeal of φ as a highly aesthetically pleasing ratio, Leonardo DaVinci's *Deluge over a city*. In this work, and in the series of sketches leading up to completion of the project, DaVinci explores the beauty and simplicity of logarithmic spirals in the form of a violent deluge that washes away every trace of the city. These spirals, however, are closely related to the Golden Ratio (and specifically Golden Triangles). Figure 4 illustrates the use of such triangles to construct a logarithmic spiral; beginning with a single Golden Triangle ΔABC, bisect one of the 72° angles and adjoin the vertex to the opposing side (i.e. construct segment BD). The smaller triangle is itself a Golden Triangle, which can be bisected and adjoined to the opposing side (via segment CE), in turn yielding a third Golden Triangle. Repeating this process (which can continue indefinitely), one can construct as many increasingly smaller Golden Triangles as time and patience permit. Then, drawing a spiral that intersects each of these constructed points in the order in which they were drawn (here, A-B-C-D-E-F-G-H), one discovers that the resulting spiral is logarithmic [3], as in *Deluge over a city*. DaVinci's work is mesmerizing and beautiful, a testament to the aesthetically appealing nature of the logarithmic spiral and the Golden Ratio contained therein.

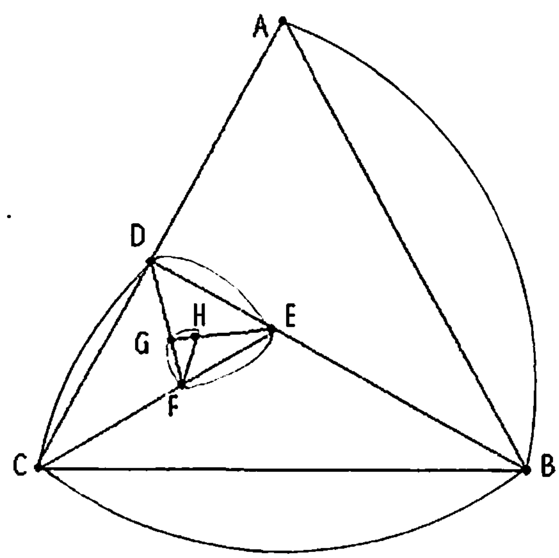

Figure 4 Constructing a logarithmic spiral from a Golden Rectangle

IV. PHI GROWS UP

Modern developments in mathematics have enabled artists, scientists, and mathematicians alike to gain a deeper understanding of the intricacies of the Golden Ratio and its astonishing properties. One of the most interesting properties of φ that has come to light only during the last two centuries is the Golden Ratio's curious continued fraction representation, a derivation of which follows. Suppose a generic line segment, illustrated in Figure 1 above, were cut in the Golden Ratio such that the small end of the segment has unit length and long end length x. Then, since the ratio of x to 1 equals the ratio of (x + 1) to x,

$$x = \frac{x+1}{x},$$

or

$$x = 1 + \frac{1}{x}.$$

One can then substitute the expression into the denominator on the left hand side to obtain

$$x = 1 + \frac{1}{1 + \frac{1}{x}}.$$

Continuing in this fashion,

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}} = [1;1,1,1,1,1,\ldots].$$

Thus, the fractional expansion for φ, denoted x above, is an infinite continued fraction in which every integer is unity. The fact that the continued fraction representation is infinite indicates that φ is an irrational number [1]; compare the ease with which this fact was established in comparison to the proof of the irrationality of φ above.

The Golden Ratio has several unique algebraic properties; by definition, $\varphi^2 = \varphi + 1$, so $\varphi^2 \sim 2.618033989$ or

$$\phi^2 = 2 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}} = [2;1,1,1,1,1,\ldots] = \phi + 1..$$

Furthermore, when the continued fraction form of φ^{-1} is calculated, one obtains

$$\frac{1}{\phi} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}} = [0;1,1,1,1,1,\ldots] = \phi - 1.$$

The continued fraction form of φ is interesting because it reveals the connection between the Golden Ratio and the Fibonacci sequence; when one computes the convergents of the continued fraction version of φ,

$$c_1 = 1 = 1/1; \quad c_2 = 2 = 2/1; \quad c_3 = 3/2; \quad c_4 = 5/3; \quad c_5 = 8/5.$$

Thus, the numerators and denominators are numbers in the Fibonacci sequence (1, 1, 2, 3, 5, 8, ...), with the numerators "one step ahead" of the denominators.

V. THE BINET FORMULA

Since the Fibonacci sequence is a sequence defined by a linear recursion formula, it must be the case that there exists some closed-form version of the sequence that allows one to compute the Fibonacci number appearing at position n without resorting to the time-consuming alternative of computing every prior number in the sequence (a daunting task even for a powerful computing algorithm, given large enough n). A closed form for the Fibonacci sequence had been pondered and sought since the series was introduced in the thirteenth century; the equation was apparently discovered by Euler in the eighteenth century and rediscovered by Jacques Binet nearly a century later [3]. The closed form of the sequence takes the form

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}.$$

The verity of this formula shall be proven by induction:

Basis Case: for n = 1, the formula in question yields

$$F(1) = \frac{\phi-(1-\phi)}{\sqrt{5}} = \frac{-1+2\cdot\phi}{\sqrt{5}} = \frac{-1+(1+\sqrt{5})}{\sqrt{5}} = 1,$$

which is indeed the first number in the Fibonacci sequence.

Induction Case: suppose the equation for F(n) is accurate for some arbitrary n, n > 1. Since the Fibonacci sequence

$$F(n+2) = F(n+1) + F(n)$$

or

$$F(n+2) - (n+1) - F(n) = 0$$

closely resembles the equation for the derivation of φ (i.e. $x^2 - x - 1 = 0$), then a sufficient verification of the Binet formula is to construct a function composed of a linear combination of the two roots of the φ equation, φ and (1 – φ), and show that this linear combination has properties and values identical to the Fibonacci sequence :

$$F'(n) = a\cdot\phi^n + b\cdot(1-\phi)^n,$$

where a and b are determined by initial conditions (i.e. plugging in known values of the Fibonacci sequence and solving for a and b). Then, for the induction (n + 1) case,

$$F'(n+1) = a\cdot\phi^{n+1} + b\cdot(1-\phi)^{n+1};$$

however, by the defining equation for φ,

$$\phi^2 = \phi + 1.$$

Multiplying through by a factor of φ^{n-1},

$$\phi^{n+1} = \phi^n + \phi^{n-1}.$$

Inserting this relation into the induction step equation above,

$$F'(n+1) = a \cdot \phi^{n+1} + b \cdot (1-\phi)^{n+1}$$
$$F'(n+1) = a \cdot (\phi^{n} + \phi^{n-1}) + b \cdot ((1-\phi)^{n} + (1-\phi)^{n-1})$$
$$F'(n+1) = a \cdot \phi^{n} + a \cdot \phi^{n-1} + b \cdot (1-\phi)^{n} + b \cdot (1-\phi)^{n-1}$$
$$F'(n+1) = a \cdot \phi^{n} + b \cdot (1-\phi)^{n} + a \cdot \phi^{n-1} + b \cdot (1-\phi)^{n-1}$$
$$F'(n+1) = (a \cdot \phi^{n} + b \cdot (1-\phi)^{n}) + (a \cdot \phi^{n-1} + b \cdot (1-\phi)^{n-1})$$
$$F'(n+1) = F'(n) + F'(n-1).$$

Thus, the equation F`(n) satisfies the recursive relationship among the numbers in the Fibonacci sequence. The Binet formula, F(n), is this equation F`(n) with the substitution

$$a = \frac{1}{\sqrt{5}}, b = \frac{-1}{\sqrt{5}}.$$

Consequently, by induction, the Binet formula gives the nth term in the Fibonacci sequence.

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