A Look at the Fibonacci Numbers

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Abstract. In this paper a brief history of the Fibonacci sequence will be given, and the recursion formula, some identities, and particular theorems involving Fibonacci sequences will be discussed and given proofs. The remaining section will look at applications and interesting occurrences of the Fibonacci sequence.

1. The History of the Fibonacci Sequence

The Fibonacci sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

and the numbers in this sequence are called Fibonacci numbers. They were named after Leonardo of Pisa who wrote under the pseudonym of Fibonacci. This infinite set of positive integers appears in the rabbit problem in his work the *Liber Abaci* (1202) which briefly stated is:

> Beginning with one pair of rabbits, how many pairs of rabbits will there be after one year if every month each pair produces another pair that is reproductive two months after their birth with the assumption that no rabbit dies?

The solution to this question lies within this sequence and thus the twelfth term, 144, corresponding to the twelfth month is the number of pairs of rabbits produced after one year.

2. The Recursive Definition of the Fibonacci Sequence

The linear recurrence equation for the Fibonacci sequence is

$$F_n = F_{n-1} + F_{n-2}$$

with $F_1 = F_2 = 1$ and the convention that $F_0 = 0$.

3. Summation Identities of the Fibonacci Sequence

1) $F_1 + F_2 + \ldots + F_{n-1} + F_n = F_{n+2} - 1$

Proof:

$$\begin{aligned}
F_1 &= F_3 - F_2 \\
F_2 &= F_4 - F_3 \\
&\dots \\
F_{n-1} &= F_{n+1} - F_n \\
F_n &= F_{n+2} - F_{n+1}
\end{aligned}$$

By adding each of these terms together, we get the desired result.

Example:

$$\begin{aligned}
F_1 + F_2 + F_3 &= F_5 - 1 \\
1 + 1 + 2 = 4 &= 5 - 1
\end{aligned}$$

2) $F_1 + F_3 + \dots + F_{2n-1} = F_{2n}$

Proof: The proof is similar to (1).

Example:

$$\begin{aligned}
F_1 + F_3 + F_5 &= F_6 \\
1 + 2 + 5 = 8 &= 8
\end{aligned}$$

3) $F_2 + F_4 + \dots + F_{2n} = F_{2n+1} - 1$

Proof: From (1),

$$F_1 + F_2 + F_3 + \dots + F_{2n} = F_{2n+2} - 1$$

When we subtract the result from (2), we get the desired result.

Example:

$$\begin{aligned}
F_2 + F_4 + F_6 &= F_7 - 1 \\
1 + 3 + 8 = 12 &= 13 - 1
\end{aligned}$$

4) $F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$

Proof:

Let $F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$ be called P(n). We will prove this by induction on n.

Base Case: (n=0) $F_0^2 = 0 = F_0 * F_1$

Inductive Step: (n≥0) P(n+1) is given by a summation that's obtained from P(n) by adding one term; suggesting we should subtract. The difference is the term F_{n+1}^2. Now we are assuming that the original P(n) summation totals $F_n F_{n+1}$ and we want to show that the new P(n+1) summation totals $F_{n+1}F_{n+2}$. So we want the difference to be $F_{n+1}F_{n+2} - F_n F_{n+1}$. The actual difference is F_{n+1}^2. So we check that $F_{n+1}^2 = F_{n+1}F_{n+2} - F_n F_{n+1}$. But this is true, since it is just the Fibonacci definition upon dividing by F_{n+1}.

Hence P(n) is true for all n≥0. ■

Example:

$$F_0^2 + F_1^2 + F_2^2 = F_2 * F_3$$
$$0^2 + 1^2 + 1^2 = 2 = 1 * 2$$

4. Fibonacci Primes and Divisibility Theorems

Definition: A Fibonacci prime is a Fibonacci number that is prime. The first few Fibonacci primes are:

2, 3, 5, 13, 89, 233, 1597, 28657, 514229, ...

To date, it is unknown if there exist an infinite number of Fibonacci primes.

Theorem 1: If m|n, then $F_m \mid F_n$.

Proof:

We will prove this by induction.

Let m|n , i.e. n=m*k where k is some integer.

Assume that $F_m * F_k \mid F_m$.

Consider $F_m * F_{k+1}$.

$$F_{m(k+1)} = F_{mk+m}$$
$$F_{mk+m} = F_{mk} - 1F_m + F_{mk}F_{m+1}$$

Since $F_{mk} - 1F_m$ is divisible by F_m, $F_{mk}F_{m+1}$ is also divisible by F_m.

Hence, $F_m \mid F_n$. ■

Theorem 2: Successive Fibonacci numbers are relatively prime.

Proof:

We will prove this by contradiction.
Assume that there exists some two successive Fibonacci numbers say F_n & F_{n+1} that have a common divisor say d, where d>1. Thus their difference $F_{n+1} - F_n = F_{n-1}$ will also be divisible by d. However, we know that $F_1 = 1$ which is clearly not divisible by d. Thus, we have reached a contradiction. Hence, successive Fibonacci numbers are relatively prime. ■

5. Binet's Formula, Fibonacci Numbers, and the Golden Ratio

Definition: The golden ratio, phi, is the positive solution to $x^2 - x - 1 = 0$, which is equal to $\varphi = (1+\sqrt{5})/2$.

In 1843, Binet discovered a formula for expressing F_n in terms of the integer n, $F_n = (\varphi^n - (1-\varphi)^n)/\sqrt{5}$.

Proof:

F_n is the nth Fibonacci number, defined by
$F_0 = 0$
$F_1 = F_2 = 1$
$F_n = F_{n-1} + F_{n-2}$
Let a= $\varphi = (1+\sqrt{5})/2$ & b=-1/a
a & b are roots of the quadratic equation $x^2 - x - 1 = 0$.
$F_n = (a^n - b^n)/(a-b)$

We will prove Binet's formula by induction.
Base Case: n=0 is trivial, $F_0 = 0 = (a^0 - b^0)/(a-b)$
n=1, $F_1 = 1 = (a^1 - b^1)/(a-b)$

Then for k≥1, assume it is true for all n≤k
(in particular for n=k and n=k-1), so that

$$F_k = (a^k - b^k)/(a-b)$$
$$F_{k-1} = (a^{k-1} - b^{k-1})/(a-b)$$

Adding these two equations together, LHS becomes F_{k+1}, according to the recursion defining the Fibonacci numbers. Rearrange the RHS into the form

$$F_{k+1} = (a^k + a^{k-1} - b^k - b^{k-1})/(a-b)$$
$$F_{k+1} = (a^{k-1} * (a+1) - b^{k-1} * (b+1))/(a-b)$$

Now using the facts that $1+a=a^2$ & $1+b=b^2$, because a & b are roots of $x^2-x-1=0$. Then

$$F_{k+1} = (a^{k-1} * (a^2) - b^{k-1} * (b^2))/(a-b)$$

$$F_{k+1} = (a^{k+1} - b^{k+1})/(a-b)$$

Which is Binet's formula for F_{k+1}.
Hence Binet's formula is true for all integers n≥0. ■

Johannes Kepler noticed that the ratio of successive Fibonacci numbers converges to the golden ratio $\varphi = (1+\sqrt{5})/2$.

<u>Proof</u>:

F_n is the nth Fibonacci number, defined by
$F_0 = 0$
$F_1 = F_2 = 1$
$F_n = F_{n-1} + F_{n-2}$
Let a=$\varphi = (1+\sqrt{5})/2$ & b=-1/a
a & b are roots of the quadratic equation $x^2-x-1=0$.
$F_n = (a^n - b^n)/(a-b)$

$$\begin{aligned} F_{n+1}/F_n &= (a^{n+1} - b^{n+1})/(a^n - b^n) \\ &= (a*a^n - a*b^n + a*b^n - b*b^n)/(a^n - b^n) \\ &= (a*(a^n - b^n) + b^n*(a-b))/(a^n - b^n) \\ &= a + (b^n*(a-b))/(a^n - b^n) \\ &= a + (a-b)/((a/b)^{n-1}) \end{aligned}$$

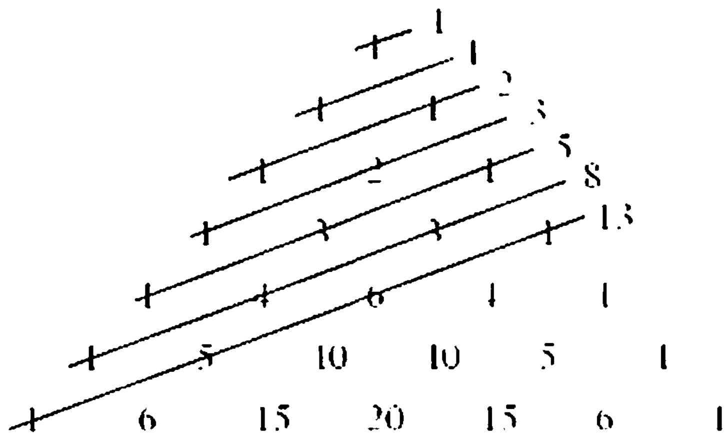
Now as n grows, $|(a/b)^n| = a^{(2n)}$ this grows without bound because a>1, so the fraction in the last equation approaches 0. That proves that F_{n+1}/F_n approaches a, as desired. ■

6. Applications and Interesting Occurrences of Fibonacci Numbers

I. The Fibonacci Numbers and Pascal's Triangle

The Fibonacci Numbers are found along the shallow diagonals of Pascal's Triangle.
<u>Example</u>:

$F_1 = 1$ corresponds to the 1st diagonal
$F_7 = 13$ corresponds to the 7th diagonal



II. The Zeckendorf Representation

<u>Theorem 3</u>: Any positive integer N can be expressed as a sum of distinct Fibonacci numbers, no two of which are consecutive; that is,

$$N = F_{k_1} + F_{k_2} + \ldots + F_{k_r}$$
$$k_1 \geq 2, \& k_{j+1} \geq k_j + 2$$
$$for$$
$$j = 1, 2, \ldots, r-1$$

III. Hilbert's Tenth problem

As stated is: Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.

Yuri Matiyasevich solved this question by showing that the Fibonacci numbers can be defined by a Diophantine equation, thus leading him to his solution.

References:

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