MATH 566 — SECOND MIDTERM EXAM

February 23, 2007

NAME: Solutions

1. (3+2+6+4 points) Let R be a binary relation on a set A.

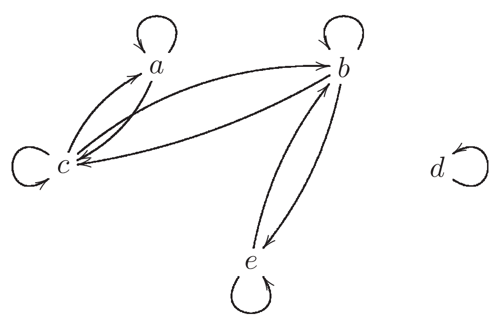
(a) Define what it means for R to be reflexive, symmetric, and transitive.

The relation R is reflexive if for every $a \in A$ one has aRa.
The relation R is symmetric if whenever $x, y \in A$ are such that xRy, then yRx.
The relation R is transitive if whenever $x, y, z \in A$ are such that xRy and yRz, then xRz.

(b) Define what it means for R' to be the transitive closure of R.

The relation R' on A is the transitive closure of a relation R if it satisfies the following two conditions:
1. $R \subseteq R'$
2. If S is a relation on A that is transitive and $R \subseteq S$, then $R' \subseteq S$.

(c) Let R be given by the following directed graph. Is R reflexive? symmetric? transitive? Be sure to give reasons.

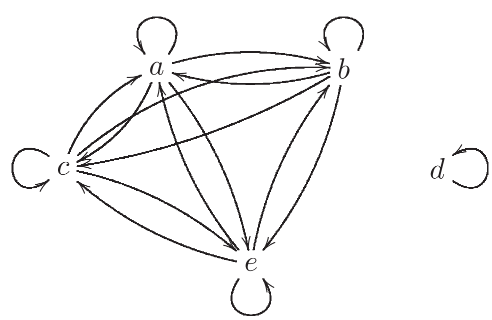


Reflexive: yes

Symmmetric: yes

Transitive: No, cRb and bRe but c is not related to e

(d) Draw the transitive closure R' of R.



2. (6+4 points)Consider the following algorithm:

for $k := 1$ to $n-1$
  for $j := 1$ to $k+1$
    $x := a[k] + b[j]$
  next j
next k

(a) Compute the actual number of elementary operations that must be performed when the algorithm segment is executed.

We construct a table to help set up the summation that gives the total number of elementary operations:

$k=1$		$k=2$			$\cdots$	$k=n-1$		
$j=1$	$j=2$	$j=1$	$j=2$	$j=3$	$\cdots$	$j=1$	$\cdots$	$j=n$
1	1	1	1	1	$\cdots$	1	$\cdots$	1

Thus we have the summation:

$$\begin{aligned} 2+3+4+\cdots+n &= (1+2+3+\cdots+n)-1 \\ &= \frac{n(n+1)}{2}-1 \\ &= \frac{1}{2}n^2+\frac{1}{2}n-1. \end{aligned}$$

(b) Find the order of the algorithm segment from among the set of power functions.

The order of $\frac{1}{2}n^2+\frac{1}{2}n-1$ is $\Theta(n^2)$.

3. (5 points each) **(a)** State in terms of an inequality what the statement "$f(n)$ is $O(1)$" means.

The statement $f(n)$ is $O(1)$ means that there exists a positive integer N and a real number $c \geq 0$ so that $|f(n)| \leq c$ for all $n \geq N$.

(b) Prove that $\sum_{k=1}^{n} \frac{1}{k^5}$ is $O(1)$.

One can use an indefinite integral for this problem. One has

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{1}{k^5} &\leq \int_2^{\infty} x^{-5} dx \\
&= \lim_{b \to \infty} \int_2^b x^{-5} dx \\
&= \lim_{b \to \infty} -\frac{1}{4}(b^{-4} - 2^{-4}) \\
&= 2^{-6}.
\end{aligned}$$

Thus, we have that

$$\sum_{k=1}^{\infty} \frac{1}{k^5} \leq 1 + \frac{1}{2^6}$$

for all n.

(c) If $f(n)$ is $\Omega(1)$ and $g(n)$ is $\Omega(1)$, is $f(n) - g(n) = 0$? Is $f(n) - g(n)$ of order $\Omega(1)$? Be sure to justify your answers with proofs or counterexamples.

Let $f(n) = 10$ and $g(n) = 2$, then $f(n)$ and $g(n)$ are both $\Omega(1)$ but clearly $f(n) - g(n) \neq 0$. Now consider the case that $f(n) = 1$ and $g(n) = n$. Then $f(n)$ and $g(n)$ are both $\Omega(1)$, but $f(n) - g(n) = 1 - n$ which is not bounded from below, so is not $\Omega(1)$.

(d) If $f(n)$ is $O(n)$ and $g(n)$ is $O(n)$, is $f(n) + g(n)$ of order $O(n)$? Be sure to justify your answer with proofs or counterexamples.

Since $f(n)$ is $O(n)$ we know there exists a constant c and a positive integer N so that $|f(n)| \leq cn$ for all $n \geq N$. Similarly, there exists a M and d so that $|g(n)| \leq dn$ for all $n \geq M$. Let $T = \max(M, N)$. Then for $n \geq T$ we have

$$\begin{aligned}
|f(n) + g(n)| &\leq |f(n)| + |g(n)| \qquad \text{(triangle inequality)} \\
&\leq cn + dn \\
&(c+d)n.
\end{aligned}$$

Thus, $f(n) + g(n)$ is $O(n)$.

4. (5 points each) A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:
(1) Rabbit pairs are not fertile during their first 3 months of life, but thereafter give birth to five new male/female pairs at the end of every month.
(2) No rabbits die.
Let s_n = the number of pairs of rabbits alive at the end of month n for each integer $n \geq 1$ and let $s_0 = 1$.

(a) Compute $s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$.

We solve this by first forming a table:

Month	0	1	2	3	4	5	6	7	8
Fertile	0	0	0	1	1	1	1	6	11
Unfertile	1	1	1	0	5	10	15	15	40

Thus, we have that $s_0 = 1 = s_1 = s_2 = s_3$, $s_4 = 6$, $s_5 = 11$, $s_6 = 16$, $s_7 = 21$ and $s_8 = 51$.

(b) Find a recurrence relation for s_n. (You may wish to check it on the data from part (a) to make sure it works!)

To find s_n we observe that we have all of the pairs of rabbits from the previous month, s_{n-1}, plus all of the pairs newly born. The number of newly born is given by $5s_{n-4}$. Thus, $s_n = s_{n-1} + 5s_{n-3}$. We can check this, for example $s_8 = 51 = s_7 + 5s_4$.

(c) How many rabbits will there be at the end of two years?

We need to compute s_{12}. Observe that we have

$$\begin{aligned} s_9 &= s_8 + 5s_5 = 51 + 55 = 106 \\ s_{10} &= s_9 + 5s_6 = 106 + 80 = 186 \\ s_{11} &= s_{10} + 5s_7 = 186 + 105 = 291 \\ s_{12} &= s_{11} + 5s_8 = 291 + 255 = 546. \end{aligned}$$

Of course, this is giving the number of pairs of rabbits, so we must multiply this by 2 to get the total number of rabbits, 1092 rabbits.

5. (5 points each) Prove the following properties:

(a) If k is an integer and x is a real number with $2^k \leq x < 2^{k+1}$, then $\lfloor \log_2 x \rfloor = k$.

Since $\log_2$ is an increasing function, we can apply $\log_2$ to the inequality $2^k \leq x < 2^{k+1}$ to obtain $k \leq \log_2 x < k+1$ where we have used the fact that $\log_2 x^b = b \log_2 x$ and $\log_2 2 = 1$. Thus, by definition of the floor function we have $\lfloor \log_2 x \rfloor = k$. ■

(b) For any odd integer $n > 1$, $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2 n \rfloor$.

Let n be an odd integer. There exists a positive integer k so that $2^k < n < 2^{k+1}$. Note that the inequalities are strict because n is assumed to be odd. Applying part (a) gives that $\lfloor \log_2 x \rfloor = k$. Since the inequality $2^k < n < 2^{k+1}$ is strict, we have that n is at least 1 more then 2^k, so we have $2^k \leq n-1 < 2^{k+1}$. Applying part (a) again we see that $\lfloor \log_2(n-1) \rfloor = k = \lfloor \log_2 x \rfloor$. ■

6. (5 points each) **(a)** Prove that $\displaystyle\sum_{k=0}^{\infty} 4kx^{4k+2} = \frac{4x^6}{(1-x^4)^2}$.

Substituting x^4 into the geometric series formula we obtain:

$$\sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}.$$

Taking the derivative of each side of this equation we obtain:

$$\sum_{k=0}^{\infty} 4kx^{4k-1} = \frac{4x^3}{(1-x^4)^2}.$$

Multiplying both sides of this equation by x^3 gives the desired result.

(b) Use part (a) (or any other valid method) to calculate $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^{4k}}$.

Plugging in $x = \frac{1}{2}$ into the series from part (a) we observe first that the series converges at $x = \frac{1}{2}$ because the geometric series converges there and then simple arithmetic gives the result.

7. (10 points) Use a recursion tree to determine the order of the recurrence

$$T(n) = T(n-2) + 15n + 2.$$

Let $n = 2k + r$ where $r = 0, 1$. The recurrence tree for $T(n)$ is given by

$$\begin{array}{c} 15n+2 \\ | \\ 15(n-2)+6 \\ | \\ 15(n-4)+6 \\ | \\ \vdots \\ | \\ 15(n-2(k-2))+2 \\ | \\ T(r+2). \end{array}$$

Thus, we have

$$\begin{aligned} T(n) &= \sum_{j=0}^{k-2} (15(n-2j)+2) + T(r+2) \\ &= 15n \sum_{j=0}^{k-2} 1 - 30 \sum_{j=0}^{k-2} j + 2 \sum_{j=0}^{k-2} 1 + T(r+2) \\ &= 15n(k-1) - 30 \left(\frac{(k-2)(k-1)}{2} \right) + 2(k-1) + T(r+2) \\ &= 15n \left(\frac{n-r}{2} - 1 \right) - 15 \left(\frac{n-r}{2} - 2 \right) \left(\frac{n-r}{2} - 1 \right) + 2 \left(\frac{n-r}{2} - 1 \right) + T(r+2) \\ &= \Theta(n^2). \end{aligned}$$

8. (10 points) You may use whatever method you find easiest (as long as it is a valid method!!) to determine the order of the recurrence

$$T(n) = 4T\left(\left\lceil\frac{n}{2}\right\rceil\right) + 2n^3 + \log_2 n^2.$$

We can use the Master Method on this recursion with $a = 4$, $b = 2$, and $f(n) = 2n^3 + \log_2 n^2$. Observe that $\log_b a = \log_2 4 = 2$. Thus we want to compare $f(n)$ with n^2. It is clear that $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon = 1$. Thus, we would like to apply part 3 of the master method. However, before we can do that we need to show that for large enough n there exists a constant $c < 1$ so that $af(n/b) \leq cf(n)$. Observe that we have

$$\begin{aligned}4f(n/2) &= n^3 + 8\log_2 n - 8 \\ &= n^3 + 2\log_2 n^2 + 2\log_2 n^2 - 8 \\ &> cf(n)\end{aligned}$$

for any $0 < c < 1$ for large enough n as $2\log_2 n^2 - 8 > 0$ for large n. In particular, we can choose $c = 1/2$ and $n \geq 4$. Thus, the master method gives that $T(n)$ is $\Theta(f(n)) = \Theta(n^3)$.