# MATH 566 — SECOND MIDTERM EXAM

## February 23, 2007

NAME: ______________________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 8 problems.
3. Do not separate the pages of the exam.
4. Your work should be neat and legible. You may and should use the back of pages for scrap work.
5. Show all your work and explain your reasoning. Partial credit will NOT be given if I cannot easily follow your logic.
6. Please turn **off** all cell phones.

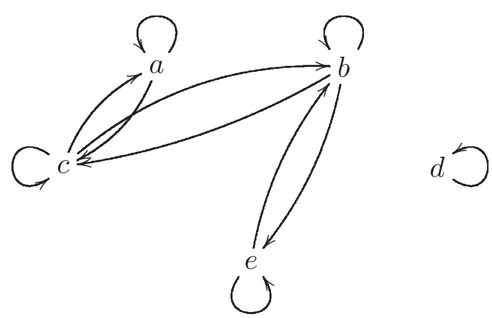
| PROBLEM | POINTS | SCORE |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 20 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| TOTAL | 100 | |

**1.** (3+2+6+4 points) Let $R$ be a binary relation on a set $A$.

**(a)** Define what it means for $R$ to be reflexive, symmetric, and transitive.

**(b)** Define what it means for $R'$ to be the transitive closure of $R$.

**(c)** Let $R$ be given by the following directed graph. Is $R$ reflexive? symmetric? transitive? Be sure to give reasons.

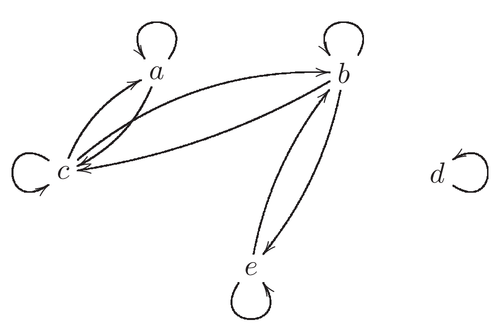



Reflexive:

Symmmetric:

Transitive:

**(d)** Draw the transitive closure $R'$ of $R$.



**2.** (6+4 points)Consider the following algorithm:

```
for k := 1 to n − 1
    for j := 1 to k + 1
        x := a[k] + b[j]
    next j
next k
```

**(a)** Compute the actual number of elementary operations that must be performed when the algorithm segment is executed.

**(b)** Find the order of the algorithm segment from among the set of power functions.

**3.** (5 points each) **(a)** State in terms of an inequality what the statement "$f(n)$ is $O(1)$" means.

**(b)** Prove that $\sum_{k=1}^{n} \frac{1}{k^5}$ is $O(1)$.

**(c)** If $f(n)$ is $\Omega(1)$ and $g(n)$ is $\Omega(1)$, is $f(n) - g(n) = 0$? Is $f(n) - g(n)$ of order $\Omega(1)$? Be sure to justify your answers with proofs or counterexamples.

**(d)** If $f(n)$ is $O(n)$ and $g(n)$ is $O(n)$, is $f(n) + g(n)$ of order $O(n)$? Be sure to justify your answer with proofs or counterexamples.

**4.** (5 points each) A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:
(1) Rabbit pairs are not fertile during their first 3 months of life, but thereafter give birth to five new male/female pairs at the end of every month.
(2) No rabbits die.
Let $s_n$ = the number of pairs of rabbits alive at the end of month $n$ for each integer $n \geq 1$ and let $s_0 = 1$.

**(a)** Compute $s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$.

**(b)** Find a recurrence relation for $s_n$. (You may wish to check it on the data from part (a) to make sure it works!)

**(c)** How many rabbits will there be at the end of two years?

**5.** (5 points each) Prove the following properties:

**(a)** If $k$ is an integer and $x$ is a real number with $2^k \leq x < 2^{k+1}$, then $\lfloor \log_2 x \rfloor = k$.

**(b)** For any odd integer $n > 1$, $\lfloor \log_2 (n-1) \rfloor = \lfloor \log_2 n \rfloor$.

**6.** (5 points each) **(a)** Prove that $\sum_{k=0}^{\infty} 4kx^{4k+2} = \frac{4x^6}{(1-x^4)^2}$.

**(b)** Use part (a) (or any other valid method) to calculate $\sum_{k=0}^{\infty} \frac{k}{2^{4k}}$.

7. (10 points) Use a recursion tree to determine the order of the recurrence

$$T(n) = T(n-2) + 15n + 2.$$

8. (10 points) You may use whatever method you find easiest (as long as it is a valid method!!) to determine the order of the recurrence

$$T(n) = 4T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 2n^3 + \log_2 n^2.$$