MATH 566 — FIRST MIDTERM EXAM SOLUTIONS

January 26, 2007

1. (6 points) Express the following sum in closed form:

$$\sum_{j=0}^{m} \binom{m}{j} (-2)^j a^{3m-3j} b^{5j}.$$

This problem is very close to homework problem #35 in section 6.7.

$$\begin{aligned}
\sum_{j=0}^{m} \binom{m}{j} (-2)^j a^{3m-3j} b^{5j} &= \sum_{j=0}^{m} \binom{m}{j} (a^3)^{m-j} (b^5)^j (-2)^j \\
&= \sum_{j=0}^{m} \binom{m}{j} (a^3)^{m-j} (-2b^5)^j \\
&= (a^3 - 2b^5)^m
\end{aligned}$$

where the last equality follows from the binomial theorem.

2. (6 points) For all integers $n \geq 0$, prove that

$$5^n = 10^n - 5n10^{n-1} + 25\binom{n}{2}10^{n-2} - \cdots + 10(-1)^{n-1}\binom{n}{n-1}5^{n-1} + (-1)^n 5^n.$$

Observe that the binomial theorem gives

$$\begin{aligned}
5^n &= (10-5)^n \\
&= \sum_{j=0}^{n} \binom{n}{j} 10^{n-j} (-5)^j \\
&= 10^n - 5n10^{n-1} + 25\binom{n}{2}10^{n-2} - \cdots + 10(-1)^{n-1}\binom{n}{n-1}5^{n-1} + (-1)^n 5^n
\end{aligned}$$

as desired.

3. (6 + 7 points) **(a)** Let x be a real number. Define $\lceil x \rceil$.

$\lceil x \rceil$ is the unique integer n such that $n - 1 < x \leq n$.

(b) Let m be an integer and x a real number that is NOT an integer. Prove that

$$\lceil m - x \rceil = m + 1 - \lceil x \rceil.$$

Proof: Let $\lceil m - x \rceil = n$. In other words, n is the unique integer so that $n - 1 < m - x \leq n$. By adding $-m$ to each side of the inequality we obtain $n - m - 1 < -x \leq n - m$. We then multiply through by -1, remembering to change the order of the inequalities to obtain $m - n \leq x < (m - n) + 1$. Since x is not an integer, we must have that $x \neq m - n$. Thus we have $m - n < x < (m - n) + 1$. In particular, we have that $\lceil x \rceil = (m - n) + 1$. Solving this for n we obtain $\lceil m - x \rceil = n = m + 1 - \lceil x \rceil$. $\blacksquare$.

4. (5 + 7 + 8 points) Normal poker is played with a 52 card deck. Each card has a rank, which can be any of 13 possibilities:

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J,Q,K,A}\}$$

and a suit, which has 4 possibilites:

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}.$$

Instead of playing with the standard deck, we want to play a new kind of poker where one gets 4 cards instead of 5 and they are chosen from a deck of cards where the cards have rank among the possibilities:

$$\{2, 3, \ldots, 31, 32\}$$

and suit among the possibilities:

$$\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit, \bigstar\}.$$

(a) How many cards are in this deck of cards?

First observe that there are $32 - 2 + 1 = 31$ different ranks. Each rank can be any of 5 different suits, so there are $31 \cdot 5 = 155$ different cards in the deck.

(b) What is the probability of getting four of a kind?

We only choose 4 cards in this type of poker, so we must choose all 4 cards of a single rank. There are 31 ranks to choose from, so we can choose a rank in $\binom{31}{1} = 31$ different ways. There are 5 possible suits to choose from, and we must choose 4 of them. There are $\binom{5}{4} = 5$ ways to accomplish this. Thus, there are $31 * 5 = 155$ different ways to get 4 of a kind. To compute the probability, we compute that there are $\binom{155}{4} = 23130030$ different hands of poker. The probability of getting 4 of a kind is

$$\frac{155}{23130030}.$$

(c) What is the probability of getting exactly one pair? (Remember, this means two cards of the same rank but no more then 2 cards can have the same rank. We also do not want to include 2 pair in this count!)

First we need to choose the rank of the pair. There are $\binom{31}{1} = 31$ ways to do this. Once we have chosen the rank of the pair, we need to choose the suits of the pair. There are $\binom{5}{2} = 10$ ways to do this. Now we need to choose the other two cards so that they are not of the same rank. Since we have already used one of the ranks, we obtain that there are $\binom{30}{2} = 435$ ways to do this. Once we have chosen the last two ranks, we just need to choose a suit for each of the cards. There are $\binom{5}{1} = 5$ ways to do this for each card. Thus, the total number of hands containing exactly one pair is

$$31 \cdot 10 \cdot 435 \cdot 5 \cdot 5 = 3371250.$$

Thus, the probability of getting exactly one pair is

$$\frac{3371250}{23130030}.$$

5. (5 points each) There are currently 28 students enrolled in this class. There are 365 days in a year (we ignore leap years!)

(a) What is the total possible number of ways that birthdays can be associated with people in a class of 28 students? For example, one way would be the first student was born on January 1, the second on January 2, etc.

There are 365^{28} ways to associate birthdays to people in the class.

(b) How many ways could birthdays be associated with the students in the class so that no two students shared the same birthday?

If we do not want students to have the same birthday, we need to choose a different birthday each step. There are

$$365 \cdot 364 \cdots (365 - 27) = \frac{365!}{(365 - 28)!}$$

ways to do this.

(c) What is the probability that at least 2 people it the class of 28 have the same birthday?

Using part (b), we can calculate the probability that no students have the same birthday. To get the probability that at least 2 share the same birthday, we just subtract the probability that no students have the same birthday from 1:

$$1 - \frac{\frac{365!}{(365-28)!}}{365^{28}} = 1 - \frac{365!}{365^{28}(365 - 28)!} = 0.65.$$

(d) What is the minimum number of students needed in a class so that the probability of at least 2 students sharing the same birthday is greater then 50%?

We can just use the formula given in part (c) with different numbers of students. Since we are over 50% with 28 students, we can just work our way down until we get under 50%. We see that for 23 students the probability is 50.7% and for 22 students the probability is 47.5%. Thus, the minimum needed is 23 students.

6. (8 points) Use the triangle inequality ($|a+b| \leq |a|+|b|$) and the fact that for $x > 1$, r, s rational numbers with $r < s$, we know that $x^r < x^s$ to prove that $f(x) = 5x^{16} + 2x^5 - 3x^2 - 1$ is $O(x^{16})$.

We assume that $x > 1$ so that we can use the inequality $x^r < x^s$ for $r, s \in \mathbb{Q}$ with $r < s$ mentioned in the problem. Thus, we have

$$\begin{aligned} |f(x)| &\leq 5|x^{16}| + 2|x^5| + 3|x^2| + 1 \qquad \text{(triangle inequality)} \\ &= 5x^{16} + 2x^5 + 3x^2 + 1 \qquad (x > 1) \\ &= 5x^{16} + 2x^{16} + 3x^{16} + x^{16} \qquad \text{(by the inequality mentioned above)} \\ &= 11x^{16}. \end{aligned}$$

Thus, $f(x)$ is $O(x^{16})$ by the definition of the big O-notation.

7. (7 points) Explain the error in the following argument that attempts to count the number of hands in poker with a **pair or better.**

> In order to form a pair, we first choose a denomination. There are $\binom{13}{1}$ ways to choose a denomination. We then choose 2 cards from that denomination. There are $\binom{4}{2}$ ways to do this. Since we now have our pair and have used only 2 cards, any other choice of three cards will work. There are $\binom{50}{3}$ ways to choose three cards from the remaining 50 cards.

The problem with this argument is not the possibility of getting 3 or 4 of a kind as we are trying to count getting a pair or better. The problem is as follows. One possible hand would be that when we choose our denomination we choose a 2. We then choose suits $\heartsuit$ and $\diamondsuit$ for our 2's. This gives our pair $2\heartsuit, 2\diamondsuit$. According to this method we are now allowed to choose any of the remaining cards to fill out our hand. So one possibility would be to choose $3\heartsuit, 3\diamondsuit, 4\clubsuit$. Thus our hand is $2\heartsuit, 2\diamondsuit, 3\heartsuit, 3\diamondsuit, 4\clubsuit$. However, another way would be to choose 3 as our denomination at the beginning and then choose $3\heartsuit, 3\diamondsuit$ as our initial pair. Then our remaining 3 cards chosen could be $2\heartsuit, 2\diamondsuit$, and $4\clubsuit$. This gives the same hand we had before, but we produce it a different way. This shows that the counting method outlined above will count the same hands more then once!

8. (5 points each) A group of 15 friends is hanging out after a grueling Friday evening midterm. They decide to go out and see a movie, but only have 1 car to drive that will fit 6 people.

(a) How many ways can a group of 6 be chosen from the friends to go to the movies?

There are $\binom{15}{6} = 5005$ ways to choose a group of 6 to go to the movies.

(b) Suppose 10 of the friends are male and 5 are female. How many groups of 6 can be chosen so that at least 2 females go to the movies?

First we calculate how many ways can we choose either 0 or 1 female, then subtract this answer from part (a) to get how many groups of 6 can be chosen with at least 2 females. To get 0 females, all 6 must be male. There are $\binom{10}{6} = 210$ ways to choose all males. To get 1 female, we need 5 males. There are $\binom{10}{5} \cdot \binom{5}{1} = 252 \cdot 5 = 1260$ ways to do this. Thus, there are $5005 - (210 + 1260) = 3535$ groups of 6 that can be chosen with at least 2 females.

(c) Suppose 1 of the guys and 1 of the girls used to be a couple and they refuse to go to the movies together. How many groups of 6 can be chosen to go to the movies?

This problem was not meant to require that 2 girls still went to the movies, but as long as you indicated that is what you were doing it was fine. The breakdown of possibilities is as follows: the male of the couple could go to the movies, the female of the couple could go to the movies, or neither could go to the movies. We need to calculate how many possibilities there are in each case.
The male goes to the movies: We need to choose 5 other people to go to the movies, but that set of 5 cannot contain his ex-girlfriend. Thus, we need to choose 5 people from a sample of 13. There are $\binom{13}{5} = 1287$ ways to do this.
The female goes to the movies: This is the same as if the male goes, so there are again 1287 ways to do this.
Neither goes to the movies: Now we need to choose 6 people from a sample of 13, so there are $\binom{13}{6} = 1716$ ways to choose groups without the couple.
Thus, there are $1287 + 1287 + 1716 = 4290$ groups of 6 that can be chosen to go to the movies.

(d) Suppose we don't have to fill up the car entirely with people, but that no one wants to go unless at least 4 people will go. Further, no one wants to go unless at least 2 girls go. Finally, the movie that has been chosen just happens to be the movie the couple in part (c) went to on their first date. Upon hearing this, they decide to get back together and insist upon going to the movie. How many groups can be chosen to go to the movies now?

We need to calculate the number of groups when 4 people go, 5 people go, and 6 people go.
4 people go to the movies: First, note that the couple always goes to the movies. Also, we need at least 2 girls. So we need to calculate how many groups we can form by adding 1 guy and 1 girl or 2 girls. There are $\binom{9}{1}$ ways to add a guy and $\binom{4}{1}$ ways to add a girl. There are $\binom{4}{2} = 6$ ways to add 2 girls. Thus, there are 42 groups of 4 that can be chosen.
5 people go to the movies: Again the couple is going, so we need to add at least 1 girl. If we add 1 girl, there are $\binom{9}{2} \cdot \binom{4}{1} = 144$ groups to form. If we add 2 girls there are $\binom{9}{1} \cdot \binom{4}{2} = 54$ groups to form. If we add 3 girls, there are $\binom{4}{3} = 4$ groups to form. Thus, there are $144+54 + 4 = 202$ groups of 5 that can be chosen.
6 people go to the movies: Again, the couple is going, so we need to add at least 1 girl again. If we add 1 girl, there are $\binom{9}{3} \cdot \binom{4}{1} = 336$ groups to form. If we add 2 girls there are $\binom{9}{2} \cdot \binom{4}{2} = 216$ groups to form. If we add 3 girls, there are $\binom{9}{1}\binom{4}{3} = 36$ groups to form. Finally, if we add 4 girls there is $\binom{4}{4} = 1$ group to form. Thus, there are $336 + 216 + 36 + 1 = 589$ groups of 6 that can be chosen.
So we have $42 + 202 + 589 = 833$ groups that can be chosen to go to the movies so that the couple goes and at least 2 girls go.