MATH 566 — FINAL EXAM

March 14, 2007

NAME: Solutions

1. (4+2 points) Let P be the set of all points in the Cartesian plane. Define a relation R on P by aRb if a and b are the same distance from the origin.

(a) Prove that R is an equivalence relation.

For a point p in the Cartesian plane, let $|p|$ be the distance from the point to the origin. It is clear that the relation is reflexive as $|p| = |p|$. If p and q are two points in the plane with pRq, then $|p| = |q|$. Thus, $|q| = |p|$ and so qRp and the relation is symmetric. Similarly, if p, q, and r are points in the plane and pRq and qRp, then $|p| = |q| = |r|$ and so pRr and the relation is transitive. Thus, the relation is an equivalence relation.

(b) Give a geometric description of the equivalence class of the point $(2, 0)$.

The points that are equivalent to $(2, 0)$ are the points that are a distance of 2 from the origin, i.e., the equivalence class of $(2, 0)$ is the circle centered at the origin of radius 2.

2. (4 points each) A computer programming team has 17 members.

(a) How many ways can a group of 8 be chosen to work on a project?

$\binom{17}{8} = 24310$

(b) Suppose 11 team members are men and 6 are women. How many groups of 6 can be chosen that contain at most 4 women?

$\binom{17}{6} - \binom{11}{0}\binom{6}{6} - \binom{11}{1}\binom{6}{5} = 12309$

(c) Suppose two team members refuse to work together on projects. How many groups of 8 can be chosen to work on a project?

$\binom{15}{8} + \binom{15}{7} + \binom{15}{7} = 19305$

(d) Suppose two team members insist on either working together or not at all on projects. How many groups of 8 can be chosen to work on a project?

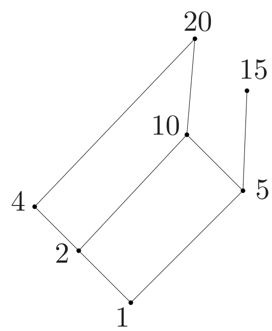
$\binom{15}{6} + \binom{15}{8} = 11440$

3. (4 points each) Let $A = \{1, 2, 4, 5, 10, 15, 20\}$ and consider the "divides" relation on this set.

(a) Define the term "antisymmetric". Is this relation antisymmetric?

A relation R is antisymmetric on a set A if whenever there are $a, b \in A$ so that aRb and bRa then $a = b$. This relation is antisymmetric.

(b) Draw the Hasse diagram for this relation.

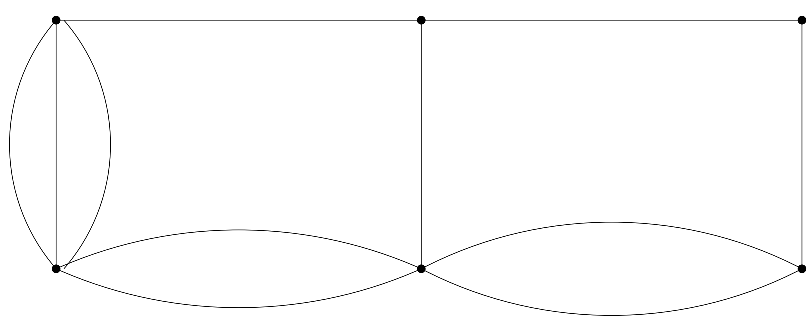


(c) Find all greatest, least, maximal, and minimal elements for the divides relation on A.

least = 1, minimal = 1, greatest = none, maximal = 15, 20

4. (4 points each) Draw a graph with the specified properties or show that no such graph exists.

(a) A graph with 6 vertices of degrees 4, 5, 3, 5, 2, and 3.



(b) A graph with 4 vertices of degrees 1, 1, 3, and 4.

Such a graph is impossible. We know the total degree of a graph, i.e., the sum of the degrees of the vertices must be even. If there were such a graph, the degree would be $1 + 1 + 3 + 4 = 9$, which is impossible.

(c) A simple graph with four vertices of degrees 1, 1, 3, and 3.

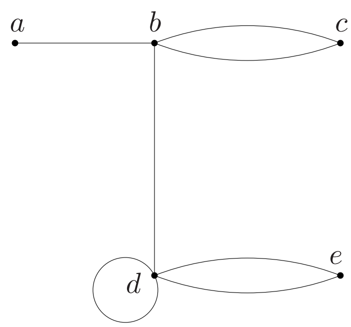
This is impossible as well. Let a, b, c, and d be the vertices of the graph. Assume without loss of generality that vertices c and d are the ones of degree 3. Since this is a simple graph, there can be no loops or parallel edges. Thus, c must have an edge connecting to a, one to b, and one to d. Likewise, d must have one connecting to a, one to b, and one to c. However, this gives that the degree of a must be at least 2.

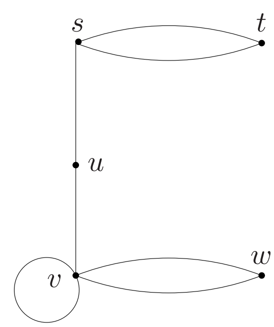
5. (4 points each) **(a)** Prove that the number of edges of a graph is an invariant for graph isomorphism.

Let G and G' be isomorphic graphs. This means there is a 1-1 correspondence $g : V(G) \longrightarrow V(G')$ and a 1-1 correspondence $h : E(G) \longrightarrow E(G')$. However, the very defintion of a 1-1 correspondence means that for every edge $e \in E(G)$ there is one and only one corresponding edge $e' \in E(G')$. Thus, there must be the same number of edges. ■

In each of the following cases, determine if the graphs are isomorphic or not. If they are, prove they are. If not, prove they are not isomorphic.

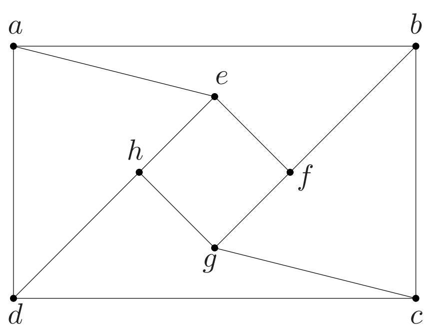
(b)

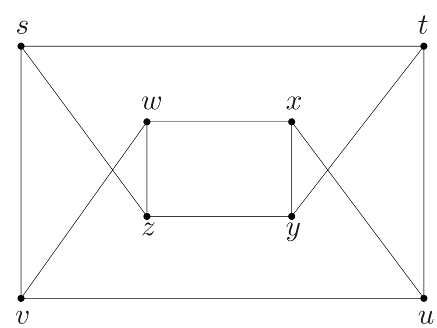




These graphs are not isomorphic. The vertex d on the first graph has degree 5 where there are no vertices of degree 5 on the second graph. Since the number of vertices is an invariant for a graph isomorphism, these graphs cannot be isomorphic.

(c)

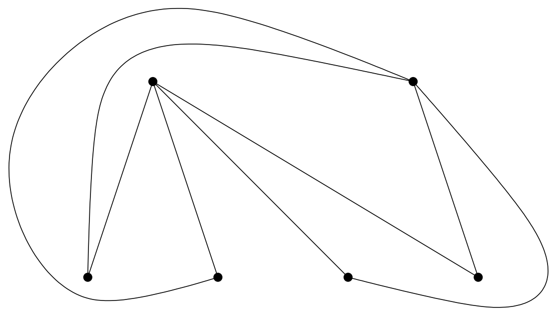




These graphs are isomorphic. If we label the first graph as G and the second graph as G', the isomorphism is given by the map $\theta : V(G) \longrightarrow V(G')$ defined by $\theta(a) = s$, $\theta(b) = t$, $\theta(c) = u$, $\theta(d) = v$, $\theta(e) = z$, $\theta(g) = x$ and $\theta(h) = w$. The function on the edges is the one corresponding to this one.

6. (4 points each) **(a)** Is it possible to connect 2 houses to 4 separate utilities so that none of the connections cross? If so, give such a connection. If not, prove it is not possible.

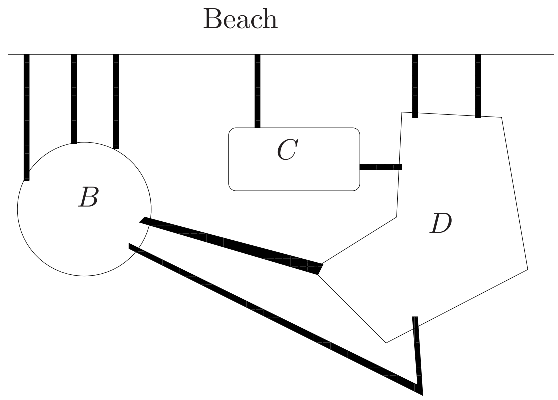
It is possible, as is illustrated by the following planar graph.



(b) Is it possible to connect 3 houses to 4 separate utilities so that none of the connections cross? If so, give such a connection. If not, prove it is not possible.

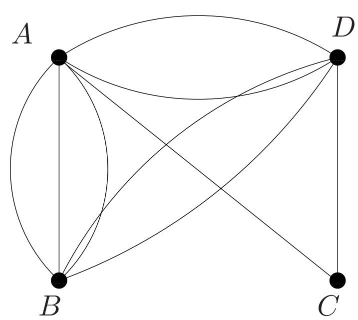
It is not possible in this case. This is equivalent to asking if $K_{3,4}$ is a planar graph. Here are a couple of proofs that it is not planar. The easiest is to observe that $K_{3,4}$ contains $K_{3,3}$ as a subgraph. If $K_{3,4}$ were planar, all of its subgraphs would have to be as well. However, we showed in class that $K_{3,3}$ is not planar.
Alternatively, one can use the formula proved in class that for a simple planar graph with no three circuits one has $e \leq 2v - 4$. Since $K_{3,4}$ is bipartite, there is no three circuit. Now just observe that $v = 7$ and $e = 12$ and so the equation is not satisfied, thus $K_{3,4}$ is not a planar graph.

7. (4 points each) Suppose you visit an ocean beach that has three islands as shown below connected by a series of bridges.



(a) Is it possible to walk along each bridge exactly once, starting and ending at the beach? If so, give such a path. If not, prove it is not possible.

We can represent the above picture with the following graph, where we have replaced "Beach" by A.



This problem is equivalent to asking if this graph has an Euler circuit. Now recall that to have an Euler circuit each vertex must have even degree. This graph does not have an Euler circuit because $\deg B = 5$.

(b) Is it possible to walk along each bridge exactly once starting on island D and ending on island B? If so, give such a path. If not, prove it is not possible.

This is possible because each of B and D have odd degree and A and C have even degree. To find such a path, one just needs to label the edges and write one down. This is fairly simple in this case and there are many correct answers.

(c) How many ways can you walk across 3 bridges, not necessarily distinct, beginning on the beach and ending on island B?

We are interested in the number of walks of length three that begin at A and end at B. The adjacency matrix of the graph is given by $A_G = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$. Thus, we want to calculate the $1, 2$ entry of A_G^3. One can calculate by hand (or using a graphing calculator) that $A_G^3 = \begin{pmatrix} 28 & 56 & 21 & 38 \\ 56 & 24 & 10 & 39 \\ 21 & 10 & 4 & 16 \\ 38 & 39 & 16 & 28 \end{pmatrix}$. Thus, the answer is there are 56 such walks.

8. (4+4+4+5+5 points) **(a)** State the formula for the finite geometric series $\sum_{k=0}^{N} x^k$.

$$\sum_{k=0}^{N} x^k = \frac{x^{N+1} - 1}{x - 1}.$$

(b) Prove that if $|x| < 1$, then $\sum_{k=0}^{n} x^k$ is $O(1)$.

To be $O(1)$ we need to show the absolute value of the sum is bounded above by a constant. Observe that we have

$$\begin{aligned} \left| \sum_{k=0}^{N} x^k \right| &\leq \sum_{k=0}^{N} |x|^k \qquad \text{(triangle inequality)} \\ &\leq \frac{1}{1 - |x|}. \end{aligned}$$

Since $\frac{1}{1-|x|}$ is a constant, we are done.

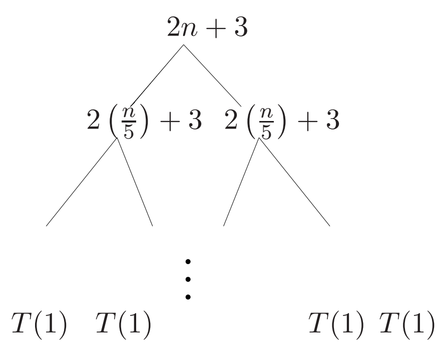
(c) What is the order of the summation $\sum_{k=0}^{\log_5 n-1} 2^k\left(2\left(\frac{n}{5^k}\right)+3\right)$? Be sure to give some justification for your answer!

$$\begin{aligned}\sum_{k=0}^{\log_5 n-1} 2^k\left(2\left(\frac{n}{5^k}\right)+3\right) &= \sum_{k=0}^{\log_5 n-1} n\frac{2^{k+1}}{5^k} + \sum_{k=0}^{\log_5 n-1} 3\cdot 2^k \\ &= 2n\sum_{k=0}^{\log_5 n-1}\left(\frac{2}{5}\right)^k + 3\sum_{k=0}^{\log_5 n-1} 2^k \\ &= \Theta(n) + 3\frac{2^{\log_5 n}-1}{2-1} \\ &= \Theta(n) + 3n^{\log_5 2} - 3 \\ &= \Theta(n) + \Theta(n^{\log_5 2}).\end{aligned}$$

Since $\log_5 2 < 1$, we see this sum is $\Theta(n)$.

(d) Use a recursion tree to estimate the order of the recursion $T(n) = 2T\left(\left\lfloor\frac{n}{5}\right\rfloor\right) + 2n + 3$.

Since we are just using the recursion tree to get an estimate, we can assume n is a power of 5 to make our lives easier and ignore the floor function. The recursion tree is given by



The i^{th} row contributes $2^i\left(2\left(\frac{n}{5^i}\right)+3\right)$. There are k rows where $n = 5^k$. Solving this for k we get that there are $\log_5 n$ rows. The recursion tree gives the estimate as the sum

$$\sum_{k=0}^{\log_5 n-1} 2^k\left(2\left(\frac{n}{5^k}\right)+3\right) + n^{\log_5 2}T(1).$$

We found in part (c) that the first part of the sum is $\Theta(n)$. We also have that $n^{\log_5 2}T(1)$ is $\Theta(n^{\log_5 2})$. Thus, the recursion tree estimates that $T(n)$ is $\Theta(n)$.

(e) Check your estimate from part (d) is correct with the Master Method.

To use the Master Method we need to compare $f(n) = 2n + 3$ with $n^{\log_b a} = n^{\log_5 2}$. We see that $f(n)$ is $\Omega(n^{\log_5 + 2 + \varepsilon})$ where we can take $\varepsilon = .01$. So now we just need to check that for large enough n there is a $c < 1$ so that $af\left(\left\lfloor \frac{n}{b} \right\rfloor\right) \leq cf(n)$. What we need is to have $\frac{2}{5}n + 6 \leq 2cn + 3c$ for n large enough. Take $c = \frac{4}{5}$. Then for $n > 3$ our condition is satisfied. Thus, the Master Method applies to give that $T(n)$ is $\Theta(f(n)) = \Theta(n)$.