MATH 454/654 — FINAL EXAM

August 4, 2010

NAME: ____

1. Do not open this exam until you are told to begin.
2. This exam has 14 pages including this cover. There are 8 problems.
3. Write your name on the top of EVERY sheet of the exam!
4. Do not separate the pages of the exam.
5. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work for each exercise so that I can see not only the answer but also how you obtained it. Be precise in your proofs!
7. Turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	20	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
8	15	
TOTAL	100	

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1. (10 points each) **(a)** Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-3)^n$.

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(b) Let $f_n(x) = \dfrac{1 + 2\cos^2(nx)}{\sqrt{n}}$. Prove that f_n converges uniformly to 0 on $\mathbb{R}$ as $n \to \infty$.

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2. (5 points each) **(a)** Find the matrix representation of the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ if T is given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + 3x_4, x_2 + x_3).$$

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(b) Calculate the limit of the sequence $\vec{x}_k = (\log(k+1) - \log(k), 2^{-k})$.

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(c) Calculate the derivative matrix $Df(1,0,1)$ for $f(x,y,z) = (x^2 - z, x - y, ze^y)$.

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3. (10 points) Prove that every convergent sequence $\{\vec{x}_k\}$ in $\mathbb{R}^n$ is bounded.

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4. (10 points) Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Show that the diagonals are perpendicular.

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5. (10 points) Let $V \subset \mathbb{R}^n$. Show that V is open if and only if there is a collection of open balls whose union is exactly V.

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6. (10 points) Let $f(x,y) = \dfrac{6x^2y^4}{x^4+y^4}$. Find the limit of $f(x,y)$ as $(x,y) \to (0,0)$, if it exists, or show that the limit does not exist.

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7. (10 points) Let $a, b \in \mathbb{R}$ with $a < b$. Let $c_1, \ldots, c_n$ be a finite set of real numbers in the interval (a, b) (you may assume $c_1 < c_2 < \cdots < c_n$.) Is $f : [a, b] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & x \in [a,b], x \notin \{c_1, \ldots, c_n\} \\ 1 & x \in \{c_1, \ldots, c_n\}. \end{cases}$$

integrable? If so, prove it is and determine the value of the integral. If not, prove it is not. (Hint: Draw pictures and choose your partition carefully!)

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8. **(a)** Let $\{b_k\}$ be a sequence of real numbers and $b \in \mathbb{R}$. Suppose there exists $M, N \in \mathbb{N}$ such that $|b_k - b| \leq M$ for all $k \geq N$. Prove that

$$\left| nb - \sum_{k=1}^{n} b_k \right| \leq \sum_{k=1}^{N} |b_k - b| + M(n - N)$$

for all $n > N$.

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(b) Let $\{b_k\}$ be a sequence converging to b as $k \to \infty$. Prove that

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \to b$$

as $n \to \infty$. (Hint: Use part (a). For $\epsilon > 0$, let M from part (a) be $\epsilon/2$.)

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(c) Let E be a nonempty subset of $\mathbb{R}$ and let $f : E \to \mathbb{R}$. Suppose that f_n is a sequence of bounded functions on E which converges to f uniformly on E. Prove that

$$\frac{f_1(x) + \cdots + f_n(x)}{n}$$

converges to $f(x)$ uniformly on E as $n \to \infty$. (Hint: Same idea as previous problem. Where does uniform convergence come in?)