Quiz 7 Solutions

1.

$$x = \sin t - t\cos t, \quad y = \cos t + t\sin t; \qquad \frac{\pi}{4} \le t \le \frac{\pi}{2}$$

is the parametric representation of a plane curve.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ without eliminating the parameter.

Since

$$\frac{dx}{dt} = \cos t - (\cos t - t\sin t) = t\sin t$$

and

$$\frac{dy}{dt} = -\sin t + (\sin t + t\cos t) = t\cos t$$

we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \cot t$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{-\csc^2 t}{t\sin t} = -\frac{1}{t\sin^3 t}$$

(b) Find the length of the curve.

The length of the curve is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{(t\sin t)^2 + (t\cos t)^2}dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} tdt = \frac{3}{32}\pi^2$$

2. A wind with velocity 45 miles per hour is blowing in the direction of 20°NW. An airplane that flies at 425 miles per hour in still air is supposed to fly straight north. Which direction should the airplane be headed, and how fast will it then be flying with respect to the ground?

Since the plane is supposed to fly straight north, the total x (i.e., East-West) component of the plane's velocity and the wind's velocity should be zero,

$$425 \cos \theta + 45 \cos(20^\circ + 90^\circ) = 0$$

Hence,

$$\theta = \cos^{-1}\left(-\frac{45\cos(110^\circ)}{425}\right) \approx 1.53 \text{ radians, or } 87.92^\circ$$

i.e., the plane should be headed 2.08°NE.

The total y component of the plane's velocity and the wind's velocity is the actual speed of the plane flying straight north in the windy air, which is

$$45 \sin(20^\circ + 90^\circ) + 425 \sin(1.53) \approx 467 \text{ mph}$$