Quiz 6 Solutions

1. Find the corresponding Cartesian equations of the following polar equations, and sketch the graph, labeling appropriately.

(a) $\theta = \dfrac{\pi}{6}$

By one of the relations between polar coordinates and Cartesian coordinates,

$$\theta = \tan^{-1}\frac{y}{x}$$

we have $\tan^{-1}\frac{y}{x} = \frac{\pi}{6}$. Since $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, the corresponding Cartesian equation is

$$y = \frac{1}{\sqrt{3}}x$$

which is a line passing through the origin, and with slope $\frac{1}{\sqrt{3}}$.

(b) $r - \cos\theta = 0$

Multiply r through the equation, because we want to use the relation $x = r\cos\theta$.

$$r^2 - r\cos\theta = 0$$

This, in Cartesian coordinates, is

$$x^2 + y^2 - x = 0.$$

Complete the sequare, we get

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4},$$

which is a circle centered at $(\frac{1}{2}, 0)$, and with radius $\frac{1}{2}$.

(d) $r^2 - 2r\cos\theta + r\sin\theta = 0$

In Cartesian coordinates, this is

$$x^2 + y^2 - 2x + y = 0.$$

Again, to see the picture, we complete the squares

$$x^2 - 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 \quad + \quad y^2 + y + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$(x-1)^2 \quad + \quad (y+\frac{1}{2})^2 = \frac{5}{4}$$

which is a circle centered at $(1, -\frac{1}{2})$, and with radius $\sqrt{\frac{5}{4}}$.

2. $r = \sin 2\theta$ is a four-leaved rose. Find the slope of the tangent line at $\theta = \frac{\pi}{6}$, and the tangent line(s) at the pole.

By the formula of the slope of tangent line in polar coordinates,

$$m = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)} = \frac{f'\sin\theta + f\cos\theta}{f'\cos\theta - f\sin\theta}$$

we have

$$m = \frac{2\cos 2\theta \sin\theta + \sin 2\theta \cos\theta}{2\cos 2\theta\cos\theta - \sin 2\theta \sin\theta}$$

Hence, the slope of the tangent line at $\theta = \frac{\pi}{6}$ is

$$\frac{2\cos\frac{\pi}{3}\sin\frac{\pi}{6} + \sin\frac{\pi}{3}\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}} = \frac{2\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}{2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{\sqrt{3}}{4}} = \frac{5}{\sqrt{3}}.$$

To find the tangent lines at the pole, we set $f(\theta) = 0$, i.e., $\sin 2\theta = 0$, and solve for θ:

$$\theta = 0, \pm\frac{\pi}{2}, \pm\frac{2\pi}{2}, \pm\frac{3\pi}{2}, \cdots\cdots$$

They represent two lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ (or, in Cartesian, $y = 0$ and $x = 0$), which are the tangent lines at the pole.