Quiz 5 Solutions

1. Use a Taylor polynomial of order 2 to approximate $\sqrt{3.9}$, and then give an upper bound for the error of the approximation.

Let $f(x) = \sqrt{x}$. Since $f(4)$ is easy to calculate and 4 is close to 3.9, we want to expand $f(x) = \sqrt{x}$ around 4 up to order 2. As $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, the polynomial $P_2(x)$ is:

$$\begin{aligned} P_2(x) &= f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 \\ &= 2 + \frac{1}{4}(x-4) + \frac{-\frac{1}{32}}{2!}(x-4)^2 \end{aligned}$$

Therefore, the approximation is

$$\begin{aligned} \sqrt{3.9} = f(3.9) &\approx P_2(3.9) = 2 + \frac{1}{4}(3.9-4) + \frac{-\frac{1}{32}}{2!}(3.9-4)^2 \\ &= 2 + \frac{1}{4}(-0.1) - \frac{1}{64}(-0.1)^2 = 1.97484375 \end{aligned}$$

And, the remainder is

$$R_2(3.9) = \frac{f'''(c)}{3!}(3.9-4)^3 = \frac{\frac{3}{8}c^{-\frac{5}{2}}}{3!} = \frac{-0.001}{16c^{\frac{5}{2}}}$$

where c is between 3.9 and 4, so the error

$$|R_2(3.9)| = \frac{0.001}{16c^{\frac{5}{2}}} \le \frac{0.001}{16(3.9)^{\frac{5}{2}}} \approx 2.08 \times 10^{-6}.$$

2. Put the following equations in standard forms. Graph your answers, labelling your graphs appropriately.

(a) $4x^2 - y^2 - 8x - 6y - 6 = 0$

Group the terms with x and y separately, and factor out the coefficients in front of x^2 and y^2,

$$4(x^2 - 2x) - (y^2 + 6y) = 6$$

then complete the squares and put the equation in standard form,

$$\begin{aligned}
4\left(x^2 - 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2\right) &- \left(y^2 + 6y + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2\right) = 6 \\
4(x-1)^2 - 4 &- (y+3)^2 + 9 = 6 \\
\frac{(x-1)^2}{(\frac{1}{2})^2} &- \frac{(y+3)^2}{1^2} = 1
\end{aligned}$$

Therefore, it is a horizontal hyperbola centered at $(1, -3)$, with vertices $(1 \pm \frac{1}{2}, -3)$ and asymptotes $y + 3 = \pm\frac{1}{\frac{1}{2}}(x - 1)$.

(b) $x^2 - 4y - 8x - 12 = 0$

Similarly, we have

$$\begin{aligned}
\left(x^2 - 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2\right) - 4y - 12 &= 0 \\
(x-4)^2 - 16 - 4y - 12 &= 0 \\
(x-4)^2 &= 4(y+7)
\end{aligned}$$

Therefore, it is a parabola opening upward with vertex $(4, -7)$.

3. Find the equations of the tangent line and the normal line of

$$\frac{x^2}{32} + \frac{y^2}{4} = 1 \qquad \text{at } (4, \sqrt{2}).$$

First, we need to find the slope of the tangent line, i.e., y' at $(4, \sqrt{2})$. Here, we use the "implicit differentiation" method, i.e., differentiate the equation directly with respect to x.

$$\frac{2x}{32} + \frac{2yy'}{4} = 0$$

(Note that we have used Chain Rule in the second term, since we are thinking y as a function of x.) Then plug in the point $(4, \sqrt{2})$,

$$\frac{8}{32} + \frac{2\sqrt{2}y'}{4} = 0,$$

so $y' = -\frac{1}{2\sqrt{2}}$, and the tangent line is

$$y - \sqrt{2} = -\frac{1}{2\sqrt{2}}(x - 4).$$

The slope of the normal line is the negative reciprocal of the slope of the tangent line, so the equation of normal line is

$$y - \sqrt{2} = 2\sqrt{2}(x - 4).$$