Quiz 4 Solutions

1. Given that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \cdots \qquad \text{when } |x| < 1$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

find the power series representation for each of the following functions

(a) $f(x) = \dfrac{x}{2+x^2}, \qquad |x| < 1$

To make use of the given power series representation of $\frac{1}{1-x}$, we rewrite $f(x)$ as

$$f(x) = \frac{x}{2} \cdot \frac{1}{1 - (-\frac{x^2}{2})}$$

Note that $|-\frac{x^2}{2}| < \frac{1}{2} < 1$ since $|x| < 1$, we have

$$\begin{aligned} f(x) &= \frac{x}{2} \cdot \left(1 + \left(-\frac{x^2}{2}\right) + \left(-\frac{x^2}{2}\right)^2 + \left(-\frac{x^2}{2}\right)^3 + \left(-\frac{x^2}{2}\right)^4 + \cdots\right) \\ &= \frac{x}{2} - \frac{x^3}{2^2} + \frac{x^5}{2^3} - \frac{x^7}{2^4} + \frac{x^9}{2^5} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{n+1}} \end{aligned}$$

(b) $g(x) = \displaystyle\int_0^x \frac{\sin t}{t} dt$

From the given power series representation of $\sin x$, we get the power series representation of $\frac{\sin t}{t}$:

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \frac{t^8}{9!} - \cdots$$

Since power can be integrated term by term, we have

$$\begin{aligned} g(x) &= \int_0^x 1dt - \int_0^x \frac{t^2}{3!}dt + \int_0^x \frac{t^4}{5!}dt - \int_0^x \frac{t^6}{7!}dt + \int_0^x \frac{t^8}{9!}dt - \cdots \\ &= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{t^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot (2n+1)!} \end{aligned}$$

2. Find the Maclaurin polynomial (i.e., Taylor polynomial centered at 0) of order 4 for

$$l(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Explain why this is almost 1 when v, compared to c, is very small.

Apply the binomial series formula

$$(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \binom{p}{4}x^4 + \cdots \qquad (\text{when} |x| < 1)$$

with $x = -\frac{v^2}{c^2}$ and $p = -\frac{1}{2}$, we get

$$\begin{aligned} l(v) &= \left(1 + \left(-\frac{v^2}{c^2}\right)\right)^{-\frac{1}{2}} \\ &= 1 + \left(\frac{-\frac{1}{2}}{1!}\right)\left(-\frac{v^2}{c^2}\right) + \left(\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}\right)\left(-\frac{v^2}{c^2}\right)^2 + \cdots \end{aligned}$$

Therefore, the Maclaurin polynormial of order 4 for $l(v)$ is

$$1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4}.$$

When v, compared to c, is very small, $\frac{v^2}{c^2}$ is very close to 0, hence $l(v)$ is almost 1.

3. True or False: Justify your answer with appropriate reasoning.

Any function has a Taylor series expansion around any given point.

FALSE. We need to evaluate the function and all its derivatives (up to any order) at the given point when expanding the function into Taylor around the point, but this is NOT always possible for a function at a given point. For example, $\frac{1}{1-x}$ is even not defined at 1, so it does not have a Taylor series expansion around 1, but it does have an expansion around 0. Another example, $|x|$ is not differentiable at 0 though it is defined there, so it does not have a Taylor series expansion around 0, but it does have an expansion around any given point other than 0.