Quiz 3 Solutions

1. Determine if the following series converge or diverge. Be sure to justify your answer.

(a) $\displaystyle\sum_{n=5}^{\infty} \frac{n^2+1}{n^2-n+1}$

Applying the n^{th} term test to this series we see that

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2+1}{n^2-n+1} &= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{1}{n^2}} \\ &= 1 \neq 0.\end{aligned}$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, the n^{th} term test tells us that the series diverges.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln n)^{\frac{1}{2}}}{n}$

We would like to apply the integral test to this series with $f(x) = \dfrac{(\ln x)^{\frac{1}{2}}}{x}$. Using a calculator one sees that for $x \geq 3$ we have that f is continuous, nonincreasing, and greater then 0. Therefore we are justified in applying the integral test and investigate the integral:

$$\begin{aligned}\int_3^{\infty} \frac{(\ln x)^{\frac{1}{2}}}{x} dx &= \lim_{b\to\infty} \int_3^{b} \frac{(\ln x)^{\frac{1}{2}}}{x} dx \\ &= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} u^{\frac{1}{2}} du \qquad (u = \ln x, du = \frac{1}{x}dx) \\ &= \lim_{b\to\infty} \frac{2}{3}((\ln b)^{\frac{3}{2}} - (\ln 3)^{\frac{3}{2}}) \\ &= \infty.\end{aligned}$$

Since the integral diverges, the integral test tells us that the series diverges as well.

2. Determine if the following series converges or diverges. Be sure to justify your answer.

$$\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$$

Our first step is to figure out the formula for a_n. One sees that this series is given by

$$\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots = \sum_{n=1}^{\infty} \frac{n}{3^n}.$$

Now that we have the form of the series, it is fairly clearly of the form one applies the ratio test to. Applying the test we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} \\ &= \lim_{n\to\infty} \frac{n+1}{n} \frac{1}{3} \\ &= \frac{1}{3}. \end{aligned}$$

Therefore, we see that our limit is less then 1, so the series must converge by the ratio test.

3. Suppose we are given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $a_n \leq b_n$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n$ converges. Explain why it is important that $a_n \geq 0$ for all n in order to apply the comparison test to conclude $\sum_{n=1}^{\infty} a_n$ converges.

The reason it is important for the a_n to be positive is so that we actually can use the b_n to have some control over the size of the a_n. For example, if we did not require the a_n to be positive, we could take $a_n = -n$ and $b_n = \frac{1}{n^2}$. In this case we would have $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, but it is clear that $\sum_{n=1}^{\infty}(-n)$ does not converge.