MATH 153 — SECOND MIDTERM EXAM

November 9, 2005

NAME: Solutions

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 8 questions.
3. Write your name on the top of EVERY sheet of the exam!
4. Do not separate the pages of the exam.
5. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work for each exercise so that the I can see not only the answer but also how you obtained it. **Include units in your answers where appropriate.**
7. You may use your calculator. However, please indicate if it is used in any significant way.
8. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
9. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	14	
2	20	
3	6	
4	8	
5	12	
6	10	
7	12	
8	18	
TOTAL	100	

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only is the statement is *always* true. No explanation is necessary.

(a) When working in polar coordinates, each point has a unique representation.

True False

(b) The Taylor series for $f(x) = x^8 + x^2 + 10$ centered at $x = 0$ is equal to $P_8(x)$, the 8^{th} degree Taylor polynomial.

True False

(c) If f is a function satisfying $f(\theta) = f(2\pi - \theta)$ for all angles θ, then $r = f(\theta)$ is symmetric with respect to the x-axis.

True False

(d) The Taylor series of a function $f(x)$ is equal to the function for all x.

True False

(e) The quantity $\frac{dr}{d\theta}\,|_{\theta=a}$ is the slope of the line tangent to the curve $r = f(\theta)$ at $\theta = a$.

True False

(f) If the tangent line to the graph $y = f(x)$ at $x = a$ is given by $y = mx + b$, then the normal line to $y = f(x)$ at $x = a$ is given by $y = -\dfrac{1}{m}x + b$.

True False

(g) The mean value theorem for f on $[a, x]$ is equivalent to the 0 degree Taylor approximation with remainder for $f(x)$ centered at $x = a$.

True False

2. (5 points each) Give the Taylor series centered at $x = 0$ of the following functions. You are allowed to use familiar series. Be sure to state where the series converge!

(a) $f(x) = xe^{x^2}$

Recall that the Taylor series for e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all x. Therefore, substituting x^2 in for x we obtain

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

for all x. Now just multiply by x to obtain

$$xe^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

for all x.

(b) $f(x) = \int_0^x \cos(t)\, dt$

The Taylor series for $\cos(t)$ is given by

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for all t. Taking the integral we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all x.

(c) $f(x) = (1-5x)^{\frac{3}{2}}$

Recall the binomial series is given by

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

and converges for $|x| < 1$. Using $p = \frac{3}{2}$ and substituting $-5x$ for x, we obtain

$$(1-5x)^{\frac{3}{2}} = 1 + \frac{3}{2}(-5x) + \frac{\frac{3}{2}(\frac{3}{2}-1)}{2}(-5x)^2 + \cdots$$

for $-1 < -5x < 1$, i.e., $-\frac{1}{5} < x < \frac{1}{5}$.

(d) $f(x) = \dfrac{1}{2-3x}$

Here we use the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$. Rewrite $f(x)$ as

$$f(x) = \frac{1}{2}\frac{1}{1-\frac{3}{2}x}.$$

Substituting $\frac{3}{2}x$ into the equation above for geometric series and multiplying by $\frac{1}{2}$ we have

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{2}x\right)^n$$

for $-1 < \frac{3}{2}x < 1$, i.e., $-\frac{2}{3} < x < \frac{2}{3}$.

3. (6 points) Find the equation of the tangent line to $\dfrac{x^2}{27} + \dfrac{y^2}{9} = 1$ at $(3, -\sqrt{6})$.

Our first step is to find the slope by finding $\frac{dy}{dx}$ via implicit differentiation:

$$\frac{2}{27}x + \frac{2}{9}y\frac{dy}{dx} = 0.$$

Substitute the point $(3, -\sqrt{6})$ in and solve for $\frac{dy}{dx}$ to obtain $\frac{dy}{dx} = \frac{1}{\sqrt{6}}$. Now we just use the point-slope formula to find the equation of the tangent line:

$$y + \sqrt{6} = \frac{1}{\sqrt{6}}(x-3).$$

4. (2 points each) Suppose the degree 6 Taylor polynomial of the function $f(x)$ centered at $x = 3$ is given by

$$P_6(x) = -7 + (x-3) - \frac{1}{2}(x-3)^2 + \frac{5}{6}(x-3)^3 - 15(x-3)^5 + (x-3)^6.$$

(a) What is $f(3)$?

For this entire problem one just uses how we defined a Taylor polynomial in terms of the derivatives of $f(x)$. Therefore, $f(3) = P_6(3) = -7$.

(b) Is the function $f(x)$ concave up or concave down at $x = 3$? Be sure to justify your answer!

Since $-\frac{1}{2} = \frac{f^{(2)}(3)}{2!}$, we see that $f^{(2)}(3) < 0$ so the function is concave down at $x = 3$.

(c) What $f^{(3)}(3)$? Be sure to justify your answer!

As above we see that $\frac{5}{6} = \frac{f^{(3)}(3)}{3!}$, i.e., $f^{(3)}(3) = 5$.

(d) What is $f^{(4)}(3)$? Be sure to justify your answer!

Since there is no power 4 terms, we can conclude that $\frac{f^{(4)}(3)}{4!} = 0$, i.e., $f^{(4)}(3) = 0$.

5. (4 points each) Consider the equation

$$9r\cos^2(\theta) + 4r\sin^2(\theta) + 8\sin(\theta) = \frac{32}{r}.$$

(a) Convert the above polar equation into rectangular coordinates. It is not necessary to put it into standard form for this part.

Mutliplying both sides of the equation by r and then making the substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$ we obtain:

$$9x^2 + 4y^2 + 8y = 32.$$

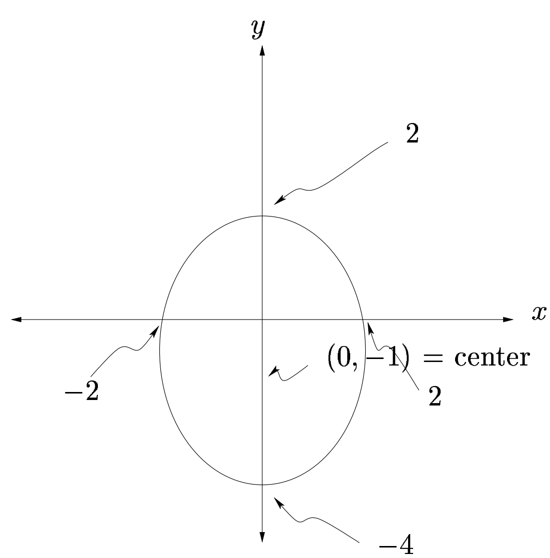
(b) Put the equation you found in part (a) into standard form and identify what familiar conic (parabola, ellipse, hyperbola, etc.) the equation represents.

We complete the square with respect to the y terms:

$$\begin{aligned} 9x^2 + 4(y^2 + 2y + 1^2) - 4(1^2) &= 32 \\ 9x^2 + 4(y+1)^2 &= 36 \\ \frac{x^2}{4} + \frac{(y+1)^2}{9} &= 1. \end{aligned}$$

This is an ellipse that has x-intercepts at $x = \pm 2$ and y-intercepts at $y = 2, -4$. Note that the $y - 1$ indicates we have shifted the center of the ellipse.

(c) Graph the equation on the set of axes below. Be sure to label and include relevant information.



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6. (3+3+4 points) Let $f(x) = \sqrt{x}$.

(a) Find $P_2(x)$ centered at $x = 9$.

Note that $f(9) = 3$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, so $f'(9) = \frac{1}{6}$, and $f^{(2)}(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ so that $f^{(2)}(9) = -\frac{1}{108}$. Therefore we have

$$P_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2.$$

(b) Use your answer to part (a) to approximate $\sqrt{9.2}$.

We just plug in 9.2 for x in part (a) to obtain the estimate:

$$\sqrt{9.2} \approx 3 + \frac{1}{6}(.2) - \frac{1}{216}(.2)^2.$$

(c) Bound the error obtained in part (b).

We need $f^{(3)}$ in order to bound the error, so observe that $f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}}$. We then know that there exists a c in $(9, 9.2)$ so that the error obtained in part (b) is equal to

$$\frac{3}{8}c^{-\frac{5}{2}}\frac{1}{3!}.2^3 = \frac{1}{2000}c^{-\frac{5}{2}}.$$

To bound the error, we look at the absolute value of the error and choose a c so as to maximize. In this case, we know that $c^{-\frac{5}{2}}$ is a decreasing function, so we have

$$|\text{Error}| \leq \frac{3}{8}9^{-\frac{5}{2}}\frac{1}{6}.2^3 = \frac{1}{486000}.$$

7. (5+3+4 points) A hydrogen atom consists of an electron of mass m, orbiting a proton, of mass M, where m is much smaller than M. The *reduced mass*, μ, of the hydrogen atom is defined by

$$\mu = \frac{mM}{m+M}.$$

(a) Express μ as m times a series in m/M.

We can factor out an M in the denominator to write

$$\mu = \frac{m}{1+\frac{m}{M}}.$$

This is merely a convergent geometric series in $-\dfrac{m}{M}$ since we have that $m << M$. Therefore we have

$$\mu = m\left(1 + \left(-\frac{m}{M}\right) + \left(-\frac{m}{M}\right)^2 + \cdots\right) = m\sum_{n=0}^{\infty}\left(-\frac{m}{M}\right)^n.$$

(b) Show that $\mu \approx m$.

For a very large M, if we approximate μ by its degree 0 Taylor polynomial we get $\mu \approx m$ from part (a).

(c) The *first order correction* to the approximation $\mu \approx m$ is obtained by including the linear term but no higher terms of the series expansion. If $m \approx M/1836$, by what percentage does including the first-order correction change the estimate $\mu \approx m$?

The description of first order correction given tells us that we should look at $\mu \approx m\left(1 - \frac{m}{M}\right)$. Observe that if there was only the "1" in the sum then there would be a 0% change. By comparison, we see that $100\% \cdot \frac{m}{M}$ gives the percentage change from the original approximation. In our situation, this is given by

$$\begin{aligned} 100\%\frac{m}{M} &= 100\%\frac{\frac{M}{1836}}{M} \\ &= 100\%\frac{1}{1836} \\ &\approx 0.054\%. \end{aligned}$$

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8. (3+5+5+5 points) Consider the equation $r^2 = 4\cos(2\theta)$.

(a) Which symmetries does this equation satisfy? Justify your answer with the appropriate tests.

We check each of the three symmetries discussed in class:
If the point (r, θ) satisfies $r^2 = 4\cos(2\theta)$, then note that $(r, -\theta)$ also satisfies the equation since $\cos(-2\theta) = \cos(2\theta)$. Therefore we have x-axis symmetry.
If the point (r, θ) satisfies $r^2 = 4\cos(2\theta)$, then note that $(-r, \theta)$ also satisfies the equation since $(-r)^2 = r^2$. Therefore we have origin symmetry.
If the point (r, θ) satisfies $r^2 = 4\cos(2\theta)$, then note that $(-r, -\theta)$ also satisfies the equation since $(-r)^2 = r^2$ and $\cos(-2\theta) = \cos(2\theta)$. Therefore we have y-axis symmetry as well.

(b) Find the equation of the tangent line to the curve at the point $\left(\sqrt{2}, \frac{\pi}{6}\right)$.

To find the equation of the tangent line we need to calculate $\frac{dy}{dx}$ at $\left(\sqrt{2}, \frac{\pi}{6}\right)$. Note that since $r > 0$, when we take the square root we are looking at the function $r = 2\sqrt{\cos(2\theta)}$. Therefore we have $y = 2\sqrt{\cos(2\theta)}\sin(\theta)$ and $x = 2\sqrt{\cos(2\theta)}\cos(\theta)$. Therefore we have

$$\frac{dy}{dt} = (\cos(2\theta))^{-\frac{1}{2}}(-2\sin(2\theta))(\sin(\theta)) + 2(\sqrt{\cos(2\theta)})\cos(\theta)$$

and

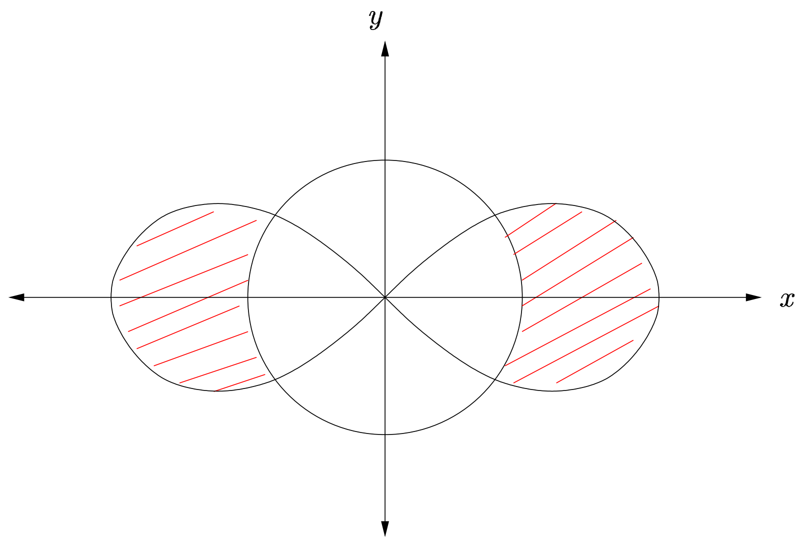
$$\frac{dx}{dt} = (\cos(2\theta))^{-\frac{1}{2}}(-2\sin(2\theta))(\cos(\theta)) - 2(\sqrt{\cos(2\theta)})\sin(\theta).$$

Substituting in $\theta = \frac{\pi}{6}$ we obtain

$$\frac{dy}{dt} = 0.$$

One can easily check that $\frac{dx}{dt} \neq 0$ so that the slope is well defined and $\frac{dy}{dx} = 0$. So we have a horizontal line for the tangent line. Substituting $\left(\sqrt{2}, \frac{\pi}{6}\right)$ into the equation $y = r\sin(\theta)$ we see that $y = \frac{\sqrt{2}}{2}$. This is the equation of our tangent line.

(c) Graph on the axes below the equations $r^2 = 4\cos(2\theta)$ and $r = 1$. Shade in the area that is inside $r^2 = 4\cos(2\theta)$ and outside $r = 1$. (Note your calculator may have difficulty with this graph, but your answer from part (a) should help you get the entire graph!)



(d) Find the shaded area from part (c).

Observe that the shaded area is just 4 times the shaded area in the first quadrant due to the symmetries we found in part (a). We need to find the intersection point. This is done by setting $r = 1$ in the equation $r^2 = 4\cos(2\theta)$. Using the calculator we get that $\cos(2\theta) = \frac{1}{4}$ when $\theta = 0.659$. Therefore our integral should go from $\theta = 0$ to $\theta = 0.659$. The shaded area is:

$$\begin{aligned}
4\int_0^{0.659} \frac{1}{2}(r^2 - 1^2)\,d\theta &= 2\int_0^{0.659} (4\cos(2\theta) - 1)\,d\theta \\
&= 4\sin(2\theta)\ |_0^{0.659} - 2\theta\ |_0^{0.659} \\
&= 4\sin(1.318) - 2(0.659) \\
&= 2.555.
\end{aligned}$$