# MATH 153 — FIRST MIDTERM EXAM

October 12, 2005

NAME: Solutions

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 8 questions.
3. Write your name on the top of EVERY sheet of the exam!
4. Do not separate the pages of the exam.
5. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so I will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work for each exercise so that the I can see not only the answer but also how you obtained it. **Include units in your answers where appropriate.**
7. You may use your calculator. However, please indicate if it is used in any significant way.
8. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
9. Please turn **off** all cell phones.

| PROBLEM | POINTS | SCORE |
|---|---|---|
| 1 | 14 | |
| 2 | 30 | |
| 3 | 5 | |
| 4 | 6 | |
| 5 | 6 | |
| 6 | 12 | |
| 7 | 12 | |
| 8 | 15 | |
| TOTAL | 100 | |

**1.** (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only is the statement is *always* true. No explanation is necessary. The correct answers are indicated in red.

**(a)** Given a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} a_n = \frac{1}{2}$, then the series diverges.

True False

**(b)** Removing a finite number of terms of a divergent series can make the series convergent.

True False

**(c)** Given sequences $\{a_n\}$ and $\{b_n\}$, one has that $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

True False

**(d)** The series $\sum_{n=1}^{100} n^2$ converges.

True False

**(e)** The series

$$1 + (x-1) + 2(x-2)^2 + 3(x-3)^3 + \cdots$$

is a power series.

True False

**(f)** If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, it is possible that the series $\sum_{n=1}^{\infty} a_n$ converges.

True False

**(g)** Suppose that the sequences $\{a_n\}$ and $\{c_n\}$ both converge to 0 and that $a_n \leq b_n \leq c_n$ for all $n \geq 100$. Then $\{b_n\}$ converges to 0 as well.

True False

**2.** (5 points each) Determine if the following converge or diverge. To receive credit your answer must be supported by valid reasoning!

**(a)** $\displaystyle\sum_{n=3}^{\infty} \frac{3n+5}{n-2}$

The $n^{\text{th}}$ term test shows this series diverges:

$$\lim_{n\to\infty} \frac{3n+5}{n-2} = 3 \neq 0.$$

**(b)** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

This is a sequence with general term given by $a_n = \dfrac{n}{n+1}$. Looking at the limit as $n \to \infty$, we see this series converges to 1.

**(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{3n^{12}+4n^7+2}{9n^{17}+3n^5+16}$

This is a good series to try the limit comparison test with. Compare this series with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$. We obtain:

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{3n^{12}+4n^7+2}{9n^{17}+3n^5+16}}{\frac{1}{n^5}} &= \lim_{n\to\infty} \frac{3n^{17}+4n^{12}+2n^5}{9n^{17}+3n^5+16} \\ &= \frac{1}{3}\end{aligned}$$

where for the last equality we have divided the top and bottom by $n^{17}$. Therefore, the limit comparison test says that $\displaystyle\sum_{n=1}^{\infty} \frac{3n^{12}+4n^7+2}{9n^{17}+3n^5+16}$ converges since $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges by $p$-series with $p = 5$.

(Continued on the next page)

**(d)** $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Let $f(x) = x^2 e^{-x^3}$. This function can be seen to be positive, continuous, and decreasing from a graph on a calculator. Therefore, we apply the integral test and look at the integral $\int_1^{\infty} f(x)dx$:

$$\begin{aligned}
\lim_{b\to\infty} \int_1^b x^2 e^{-x^3} dx &= \lim_{b\to\infty} \int_{-1}^{-b^3} -\frac{1}{3} e^u du \qquad\qquad (u = -x^3, du = -3x^2 dx) \\
&= \lim_{b\to\infty} \left(-\frac{1}{3}\right)(e^{-b^3} - e^{-1}) \\
&= \frac{1}{3} e^{-1}.
\end{aligned}$$

Since the improper integral converges, the integral comparison test tells us the series converges as well.

**(e)** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - n}{n!}$

We begin with a simple comparison to see that

$$\sum_{n=1}^{\infty} \frac{5^n - n}{n!} < \sum_{n=1}^{\infty} \frac{5^n}{n!}.$$

Now we investigate $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$ using the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} &= \lim_{n\to\infty} \frac{5^{n+1} n!}{5^n (n+1)!} \\
&= \lim_{n\to\infty} \frac{5}{n+1} \\
&= 0.
\end{aligned}$$

Therefore, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$ converges. Now, noting that $\frac{5^n - n}{n!} > 0$ for all $n$, we see that the original series converges by the comparison test.

**(f)** $\displaystyle\sum_{k=1}^{\infty} \left(2\left(\frac{1}{3}\right)^{k-2} - \left(\frac{4}{5}\right)^k\right)$

Begin by observing that $\displaystyle\sum_{k=1}^{\infty} 2\left(\frac{1}{3}\right)^{k-2}$ is a convergent geometric series with $a = 2\left(\frac{1}{3}\right)^{-1}$ and $r = \frac{1}{3}$ and $\displaystyle\sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k$ is a convergent geometric series with $a = \frac{4}{5}$ and $r = \frac{4}{5}$. Since each of these two series is convergent, we are justified in splitting up the original series to read

$$\sum_{k=1}^{\infty} \left(2\left(\frac{1}{3}\right)^{k-2} - \left(\frac{4}{5}\right)^k\right) = \sum_{k=1}^{\infty} 2\left(\frac{1}{3}\right)^{k-2} - \sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k.$$

Since it is a difference of two convergent series, it too must be a convergent series.

**3.** (5 points) Give an example of a series that is conditionally convergent but not absolutely convergent. Justify your answer.

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is an example of such a series. We know that the series does not converge absolutely because the harmonic series diverges. However, the series converges conditionally by the alternating series test because $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n} > \frac{1}{n+1} > 0$.

**4.** (6 points) What is the convergence set for the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$. (Don't forget to check the endpoints!!)

We begin by applying the ratio test to this series:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n3^n}} \right| &= \lim_{n\to\infty} \frac{n|x|}{3(n+1)} \\ &= \frac{|x|}{3}. \end{aligned}$$

Therefore, by the ratio test we know that the series converges for $-3 < x < 3$. We now need to check the endpoints $x = \pm 3$. For $x = 3$, the series is

$$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

so it is the divergent harmonic series. For $x = -3$, the series is

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is a converging alternating harmonic series. Thus, the convergence set is given by $[-3, 3)$.

**5.** (3 points each) At your engineering firm a colleague brings you the series

$$\sum_{k=0}^{\infty} 700\left(-\frac{1}{4}\right)^k \frac{7^k}{k!}$$

and tells you that it represents the energy (in Joules) at time $t = 7$ seconds of an object she is studying undergoing dampened harmonic motion.

**(a)** Show this series converges.

Applying the ratio test we get:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{700\left(-\frac{1}{4}\right)^{n+1}7^{n+1}}{(n+1)!}\cdot\frac{n!}{700\left(-\frac{1}{4}\right)^{n}7^{n}}\right| &= \lim_{n\to\infty}\frac{7}{4}\cdot\frac{1}{n+1}\\ &= 0.\end{aligned}$$

Therefore the series converges by the ratio test since the limit is less then 1.

**(b)** Use the first 5 terms of the series to give your colleague an approximation of the energy of her object at $t = 7$ seconds. (**Don't forget the units!**)

The first 5 terms give the following approximation:

$$700 + 700\left(-\frac{1}{4}\right)\frac{7}{1!} + 700\left(-\frac{1}{4}\right)^2\frac{7^2}{2!} + 700\left(-\frac{1}{4}\right)^3\frac{7^3}{3!} + 700\left(-\frac{1}{4}\right)^4\frac{7^4}{4!} \approx 195.17\text{Joules}.$$

**6.** (3 points each) Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges when $x = -4$ and diverges when $x = 7$. Which of the following are true, false, or not possible to determine? Make sure to justify your answers.

**(a)** The power series diverges when $x = 1$.

Since the series is centered at $x = 0$ and is convergent at $x = 4$, we know the convergence set is at least $(-4, 4)$. Therefore this statement is false since we know the series converges at $x = 1$.

**(b)** The power series converges when $x = 10$.

Since the series is centered at $x = 0$ and diverges at $x = 7$, we know the radius of convergence must be less then 7. Therefore, the series must diverge at $x = 10$ so the statement is false.

**(c)** The power series converges when $x = 3$.

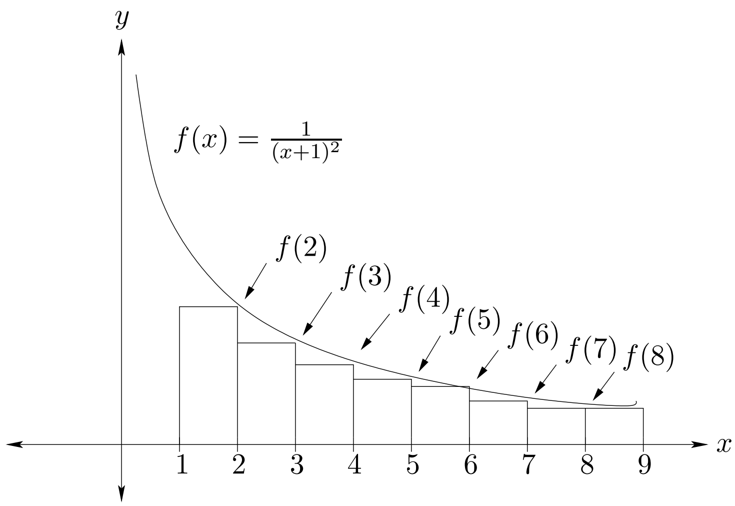
True, see explanation for Part (a).

**(d)** The power series diverges when $x = 6$.

This point falls in the interval where we don't know what is going on, so we don't have enough information to tell.

**7.** (3+3+2+4) For this problem we consider the infinite sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n+1)^2}$.

**(a)** On the following set of axes draw a picture representing this sum as well as an integral that gives an upper bound on this series. Be sure to label your picture appropriately!




**(b)** Show that this series converges.

One could use the integral test here, as suggested by the picture above. Alternatively, we can just use a simple comparison to get:

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} < \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Since $0 < \frac{1}{(n+1)^2}$ for all $n$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by $p$-series, we have that the original series converges by comparison test.

**(c)** Determine an approximation to this series by adding up the first 5 terms.

The approximation using the first 5 terms is

$$\frac{1}{(2+1)^2} + \frac{1}{(3+1)^2} + \frac{1}{(4+1)^2} + \frac{1}{(5+1)^2} + \frac{1}{(6+1)^2} \approx 0.2618.$$

**(d)** Write a series that gives the **exact** error obtained when approximating the series by the first 5 terms **and** write an integral giving an upper bound on the error. You do NOT need to evaluate this integral!

If we use the first 5 terms of a series to approximate it, the rest of the terms of the series comprise the error, which in this case is

$$\sum_{n=7}^{\infty} \frac{1}{(n+1)^2}.$$

A bound for the error is given by an integral. As seen in the picture, note that the integral starts at $x = 6$ even though the sum starts at $n = 7$. The bound on the error therefore is

$$\int_6^{\infty} \frac{1}{(x+1)^2} dx.$$

**8.** (3 points each) On the top of a 100 ft tall building you decide it would be great fun to drop a super ball. After the super ball strikes the ground, it rebounds to $\frac{6}{7}$ of its original height. Each time it returns to the ground it bounces to $\frac{6}{7}$ of its previous height. (**Don't forget the units!**)

**(a)** Let $h_n$ be the height the ball bounces to after its $n^{\text{th}}$ time hitting the ground. Find $h_1$, $h_2$, $h_3$, $h_n$.

Note that $h_1$ is the height it rebounds to after it hits the ground the first time, so $h_1 = \frac{6}{7}(100)$ ft. The second bounce will be $\frac{6}{7}$ of the first one, i.e., $h_2 = \left(\frac{6}{7}\right)\left(\frac{6}{7}100\right) = \left(\frac{6}{7}\right)^2 100$ ft. Similarly, we get $h_3 = \left(\frac{6}{7}\right)^3 100$ ft and $h_n = \left(\frac{6}{7}\right)^n 100$ ft.

**(b)** If the ball were to continue bouncing forever, what would the height of bounces approach?

We just need to see what the limit of the sequence $\{h_n\}$ is as $n \to \infty$. This is given by

$$\lim_{n\to\infty} h_n = 0 \text{ ft.}$$

**(c)** Let $d_n$ be the distance the ball has travelled when it hits the ground for the $n^{\text{th}}$ time. For example, $d_1$ would be how far the super ball has travelled after hitting the ground for the first time. Find $d_1$, $d_2$, $d_3$, $d_n$. (A picture may be helpful!)

On the first trip to the ground the ball travels 100 ft, so $d_1 = 100$ ft. After it hits the ground, it travels up $\left(\frac{6}{7}\right)100$ ft and then travels back down the same distance. Therefore, $d_2 = 100 + 2\left(\frac{6}{7}\right)100$ ft. After it hits a second time, it travels up $\left(\frac{6}{7}\right)^2 100$ ft and down the same distance. Therefore, $d_3 = 100 + 2\left(\frac{6}{7}\right)100 + 2\left(\frac{6}{7}\right)^2 100$ ft. Similarly, we get

$$d_n = 100 + 2\left(\frac{6}{7}\right)100 + 2\left(\frac{6}{7}\right)^2 + \cdots + 2\left(\frac{6}{7}\right)^{n-1} 100 \text{ ft.}$$

(Continued on the next page)

**(d)** Find a closed form expression for $d_n$.

This is not quite exactly a geometric series as the first term does not fit the pattern. However, the rest of the series is a geometric series with $a = 200\left(\frac{6}{7}\right)$ and $r = \frac{6}{7}$. Therefore, our formula for a finite geometric series gives us

$$d_n = 100 + 200\left(\frac{6}{7}\right)\frac{1-\left(\frac{6}{7}\right)^n}{1-\left(\frac{6}{7}\right)} \text{ ft.}$$

**(e)** If the super ball were to continue bouncing forever, what would be the distance it travelled?

The ball bouncing forever is the situation where $n \to \infty$. Taking this limit in the answer to Part (d), we see that the ball would travel

$$\begin{aligned} d_{\text{total}} &= 100 + 200\left(\frac{6}{7}\right)\frac{1}{1-\left(\frac{6}{7}\right)} \\ &= 1300 \text{ ft.} \end{aligned}$$